Classical Mechanics

STUDENT SOLUTIONS MANUAL

John R. Taylor

Student Solutions Manual

to accompany

JOHN R. TAYLOR

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Preface

The decision to publish this solutions manual was not an easy one. The difficulty is this: Probably the single best way to fully understand a subject like mechanics is to solve lots of problems—*really* solve them—*on your own* (perhaps with the help of a few hints). But a solutions manual like this is an almost irresistible temptation for you, the student, to cheat yourself by peeking at my solution before really grappling with the problem. (Believe me, to read someone else's solution first really is to cheat yourself; it is an order of magnitude less effective than solving the problem yourself.) This has been the main argument against publishing a solutions manual.

On the other hand, it is exceedingly frustrating if, after making a serious effort to solve a problem yourself, you reach a point where you can't get any further and have nowhere to turn for help. Even if you do successfully complete your solution, it would be good to have a supposedly definitive one to compare with yours. It is to meet these two needs that I have decided to prepare this manual, with the compromise of including only the odd-numbered problems.

So, in the resulting book you will find the following: For every odd-numbered problem we have included the statement of the problem (lifted straight from the main text) and then its solution. The solutions are fairly complete, though I have omitted a few steps of algebra. When an odd-numbered problem depends heavily on a neighboring even-numbered problem, we have included the statement (but usually not the solution) of the latter.

Occasionally it was necessary to number an equation used in a solution; in this case we used the letter "S" followed by an appropriate number. Thus (S2.4) refers to an equation first introduced in this solutions manual (specifically the fourth such equation in Chapter 2), while (2.89) refers as always to an equation found and numbered in the main text. Similarly, figures introduced in a solution are distinguished from those in the main text by a letter "S". Thus Figure S5.1 is a figure used in the solution to Problem 5.1.

What remains now is to implore you, the reader, not to cheat yourself by peeking at my solutions before making a serious try yourself. Perhaps when you start a problem, you should put this manual somewhere out of reach! And if you do come on a roadblock, only read my solution far enough to see how to move on, and then continue on your own. Then when you finally do solve the problem, enjoy comparing your solution with mine.

Meanwhile, I hope the manual helps you to enjoy using my book. And if you find a mistake, or a better solution than mine, please let me know.

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PART I

ESSENTIALS

CHAPTER 1

Newton's Laws of Motion

PROBLEMS AND SOLUTIONS

The problems for each chapter are arranged according to section number. A problem listed for a given section requires an understanding of that section and earlier sections, but not of later sections. Within each section problems are listed in approximate order of difficulty. A single star (⋆) indicates straightforward problems involving just one main concept. Two stars (⋆⋆) identify problems that are slightly more challenging and usually involve more than one concept. Three stars (⋆⋆⋆) indicate problems that are distinctly more challenging, either because they are intrinsically difficult or involve lengthy calculations. Needless to say, these distinctions are hard to draw and are only approximate.

Problems that need the use of a computer are flagged thus: [Computer]. These are mostly classified as ⋆⋆⋆ on the grounds that it usually takes a long time to set up the necessary code — especially if you're just learning the language.

SECTION 1.2 Space and Time

1.1 ⋆ Given the two vectors $\mathbf{b} = \hat{\mathbf{x}} + \hat{\mathbf{y}}$ and $\mathbf{c} = \hat{\mathbf{x}} + \hat{\mathbf{z}}$ find $\mathbf{b} + \mathbf{c}$, $5\mathbf{b} + 2\mathbf{c}$, $\mathbf{b} \cdot \mathbf{c}$, and $\mathbf{b} \times \mathbf{c}$.

$$\mathbf{b} + \mathbf{c} = 2\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}, \quad 5\mathbf{b} + 2\mathbf{c} = 7\hat{\mathbf{x}} + 5\hat{\mathbf{y}} + 2\hat{\mathbf{z}}, \quad \mathbf{b} \cdot \mathbf{c} = 1, \quad \mathbf{b} \times \mathbf{c} = \hat{\mathbf{x}} - \hat{\mathbf{y}} - \hat{\mathbf{z}}.$$

1.3 ⋆ By applying Pythagoras' theorem (the usual two-dimensional version) twice over, prove that the length r of a three-dimensional vector $\mathbf{r} = (x, y, z)$ satisfies $r^2 = x^2 + y^2 + z^2$.

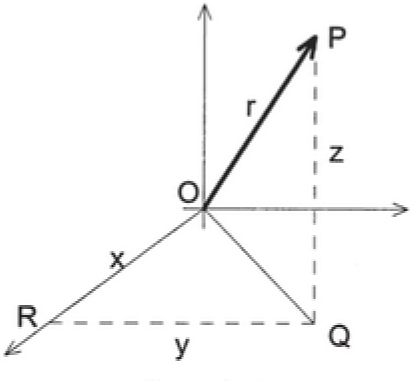

Figure S1.3

Let P be the point with position vector $\mathbf{r} = (x, y, z)$. Let Q be the projection of P onto the xy plane at $(x, y, 0)$, and let R be the projection of Q onto the x axis at $(x, 0, 0)$. Applied to the right triangle ORQ, Pythagoras' theorem tells us that

$$(OQ)^2 = x^2 + y^2,$$

and from the right triangle OQP we find

$$r^2 = (OQ)^2 + z^2 = x^2 + y^2 + z^2,$$

as required.

1.4★ One of the many uses of the scalar product is to find the angle between two given vectors. Find the angle between the vectors $\mathbf{b} = (1, 2, 4)$ and $\mathbf{c} = (4, 2, 1)$ by evaluating their scalar product.

1.5★ Find the angle between a body diagonal of a cube and any one of its face diagonals. [*Hint:* Choose a cube with side 1 and with one corner at O and the opposite corner at the point (1, 1, 1). Write down the vector that represents a body diagonal and another that represents a face diagonal, and then find the angle between them as in Problem 1.4.]

Let P be the corner at (1, 1, 1) and Q the corner below P at (1, 1, 0). Then OP is a body diagonal and OQ is a face diagonal, so the angle between OP and OQ is the required angle θ. Thus

$$\overrightarrow{OP} \cdot \overrightarrow{OQ} = \begin{cases} = |\overrightarrow{OP}|\,|\overrightarrow{OQ}| \cos\theta = \sqrt{3}\sqrt{2}\cos\theta \\ = (1, 1, 1) \cdot (1, 1, 0) = 2 \end{cases}$$

Equating these two, we find $\cos\theta = \sqrt{2/3}$ and hence $\theta = \arccos\sqrt{2/3} = 0.615$ rad or 35.3°.

1.7★ Prove that the two definitions of the scalar product $\mathbf{r} \cdot \mathbf{s}$ as $rs\cos\theta$ (1.6) and $\sum r_i s_i$ (1.7) are equal. One way to do this is to choose your x axis along the direction of $\mathbf{r}$. [Strictly speaking you should first make sure that the definition (1.7) is independent of the choice of axes. If you like to worry about such niceties, see Problem 1.16.]

If we choose our x axis in the direction of $\mathbf{r}$, then $\mathbf{r} = (r, 0, 0)$, whereas $\mathbf{s} = (s_x, s_y, s_z)$. As usual $s_x = s\cos\theta$, where θ is the angle between $\mathbf{s}$ and the x axis; with our choice of axes, this means that θ is the angle between $\mathbf{s}$ and $\mathbf{r}$. Thus, according to definition (1.7)

$$\mathbf{r} \cdot \mathbf{s}\,[\text{definition (1.7)}] = \sum r_i s_i = r s_x + 0 + 0 = rs\cos\theta = \mathbf{r} \cdot \mathbf{s}\,[\text{definition (1.6)}]$$

1.9★ In elementary trigonometry, you probably learned the law of cosines for a triangle of sides a, b, and c, that $c^2 = a^2 + b^2 - 2ab\cos\theta$, where θ is the angle between the sides a and b. Show that the law of cosines is an immediate consequence of the identity $(\mathbf{a} + \mathbf{b})^2 = a^2 + b^2 + 2\mathbf{a} \cdot \mathbf{b}$.

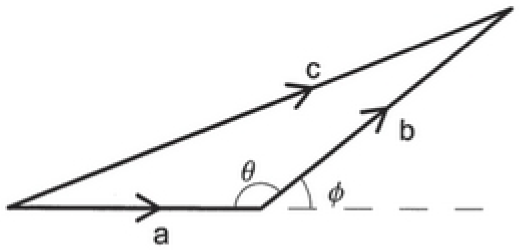

Figure S1.9

Consider the three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c} = \mathbf{a} + \mathbf{b}$ defined in the figure. The angle ϕ is the angle between $\mathbf{a}$ and $\mathbf{b}$, and the angle θ of the triangle is $\theta = \pi - \phi$. By the given identity, $c^2 = (\mathbf{a} + \mathbf{b})^2 = a^2 + b^2 + 2\mathbf{a} \cdot \mathbf{b} = a^2 + b^2 + 2ab\cos\phi = a^2 + b^2 - 2ab\cos\theta$, since $\theta = \pi - \phi$ and hence $\cos\phi = -\cos\theta$.

1.11 ⋆ The position of a moving particle is given as a function of time t to be

$$\mathbf{r}(t) = \hat{\mathbf{x}}b\cos(\omega t) + \hat{\mathbf{y}}c\sin(\omega t),$$

where b, c, and ω are constants. Describe the particle's orbit.

The particle's coordinates are $x(t) = b\cos(\omega t)$, $y(t) = c\sin(\omega t)$, and $z(t) = 0$. It remains in the plane $z = 0$ at all times, and, since

$$\frac{x^2}{b^2} + \frac{y^2}{c^2} = \cos^2(\omega t) + \sin^2(\omega t) = 1$$

it moves in an ellipse with semimajor and semiminor axes b and c in the xy plane. It is easy to see that it moves counterclockwise and returns to its starting point in a period $\tau = 2\pi/\omega$.

1.13 ⋆ Let $\mathbf{u}$ be an arbitrary fixed unit vector and show that any vector $\mathbf{b}$ satisfies

$$b^2 = (\mathbf{u}\cdot\mathbf{b})^2 + (\mathbf{u}\times\mathbf{b})^2.$$

Explain this result in words, with the help of a picture.

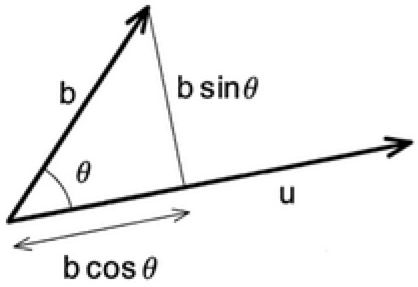

Figure S1.13

If θ is the angle between $\mathbf{b}$ and the unit vector $\mathbf{u}$, then $\mathbf{u}\cdot\mathbf{b} = b\cos\theta$ is the component of $\mathbf{b}$ in the direction of $\mathbf{u}$. Similarly, $|\mathbf{u}\times\mathbf{b}| = b\sin\theta$ is the component of $\mathbf{b}$ perpendicular to $\mathbf{u}$. Thus the result $b^2 = (\mathbf{u}\cdot\mathbf{b})^2 + (\mathbf{u}\times\mathbf{b})^2$ is simply a statement of Pythagoras' theorem.

1.15 ⋆ Show that the definition (1.9) of the cross product is equivalent to the elementary definition that $\mathbf{r}\times\mathbf{s}$ is perpendicular to both $\mathbf{r}$ and $\mathbf{s}$, with magnitude $rs\sin\theta$ and direction given by the right-hand rule. [*Hint:* It is a fact (though quite hard to prove) that the definition (1.9) is independent of your choice of axes. Therefore you can choose axes so that $\mathbf{r}$ points along the x axis and $\mathbf{s}$ lies in the xy plane.]

With the proposed choice of axes, $\mathbf{r} = (r, 0, 0)$ and $\mathbf{s} = (s_x, s_y, 0)$ where $s_y = s\sin\theta$. (To be definite we can choose the y axis so that $\mathbf{s}$ lies in the upper half of the xy plane, so that $s_y \geq 0$.) Now

$$\mathbf{r}\times\mathbf{s} = (r, 0, 0)\times(s_x, s_y, 0) = (0, 0, rs_y) = (rs\sin\theta)\hat{\mathbf{z}}.$$

This says that $\mathbf{r}\times\mathbf{s}$ is perpendicular to both of $\mathbf{r}$ and $\mathbf{s}$, has magnitude $rs\sin\theta$, and has direction given by the right-hand rule.

1.17 ★★ **(a)** Prove that the vector product $\mathbf{r} \times \mathbf{s}$ as defined by (1.9) is distributive; that is, that $\mathbf{r} \times (\mathbf{u} + \mathbf{v}) = (\mathbf{r} \times \mathbf{u}) + (\mathbf{r} \times \mathbf{v})$. **(b)** Prove the product rule

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{s}) = \mathbf{r} \times \frac{d\mathbf{s}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{s}.$$

Be careful with the order of the factors.

(a) Let us start with the x component of $\mathbf{r} \times (\mathbf{u} + \mathbf{v})$. From the definition (1.9), we see that

$$\begin{aligned}[\mathbf{r} \times (\mathbf{u} + \mathbf{v})]_x &= r_y(u_z + v_z) - r_z(u_y + v_y) = (r_y u_z - r_z u_y) + (r_y v_z - r_z v_y) \\ &= (\mathbf{r} \times \mathbf{u})_x + (\mathbf{r} \times \mathbf{v})_x.\end{aligned}$$

Since the y and z components follow in the same way, we conclude that $\mathbf{r} \times (\mathbf{u} + \mathbf{v}) = \mathbf{r} \times \mathbf{u} + \mathbf{r} \times \mathbf{v}$.
(b) Starting again from (1.9), we find for the x component

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{s})_x = \frac{d}{dt}(r_y s_z - r_z s_y) = \left(r_y \frac{ds_z}{dt} - r_z \frac{ds_y}{dt}\right) + \left(\frac{dr_y}{dt} s_z - \frac{dr_z}{dt} s_y\right) = \left(\mathbf{r} \times \frac{d\mathbf{s}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{s}\right)_x.$$

This is the x component of the desired identity. Since the y and z components follow in exactly the same way, our proof is complete.

1.19 ★★ If $\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}$ denote the position, velocity, and acceleration of a particle, prove that

$$\frac{d}{dt}[\mathbf{a} \cdot (\mathbf{v} \times \mathbf{r})] = \dot{\mathbf{a}} \cdot (\mathbf{v} \times \mathbf{r}).$$

$$\frac{d}{dt}[\mathbf{a} \cdot (\mathbf{v} \times \mathbf{r})] = \frac{d\mathbf{a}}{dt} \cdot (\mathbf{v} \times \mathbf{r}) + \mathbf{a} \cdot \frac{d}{dt}(\mathbf{v} \times \mathbf{r}) = \dot{\mathbf{a}} \cdot (\mathbf{v} \times \mathbf{r}) + \mathbf{a} \cdot (\dot{\mathbf{v}} \times \mathbf{r} + \mathbf{v} \times \dot{\mathbf{r}}).$$

The final term $\mathbf{a} \cdot (\mathbf{v} \times \dot{\mathbf{r}})$ is zero because $\dot{\mathbf{r}} = \mathbf{v}$ and $\mathbf{v} \times \mathbf{v} = 0$. The second to last term is $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{r}) = \mathbf{0}$, because $\mathbf{a} \times \mathbf{r}$ is perpendicular to $\mathbf{a}$, so their scalar product is zero. This leaves us with the requested identity.

1.21 ★★ A parallelepiped (a six-faced solid with opposite faces parallel) has one corner at the origin O and the three edges that emanate from O defined by vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. Show that the volume of the parallelepiped is $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

The base of the parallelepiped is a parallelogram with sides $\mathbf{b}$ and $\mathbf{c}$. By Problem 1.18, the area of this base is $|\mathbf{b} \times \mathbf{c}|$. The vector $\mathbf{b} \times \mathbf{c}$ is normal to the base, so if θ is the angle between $\mathbf{a}$ and this normal,

$$|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = a|\mathbf{b} \times \mathbf{c}|\,|\cos\theta| = (\text{area of base})\,|a\cos\theta|.$$

Now, $|a\cos\theta|$ is the height of the parallelepiped. Therefore $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$ is just "(area of base) × height," which is the volume, as claimed.

1.23 ★★ The unknown vector $\mathbf{v}$ satisfies $\mathbf{b} \cdot \mathbf{v} = \lambda$ and $\mathbf{b} \times \mathbf{v} = \mathbf{c}$, where λ, $\mathbf{b}$, and $\mathbf{c}$ are fixed and known. Find $\mathbf{v}$ in terms of λ, $\mathbf{b}$, and $\mathbf{c}$.

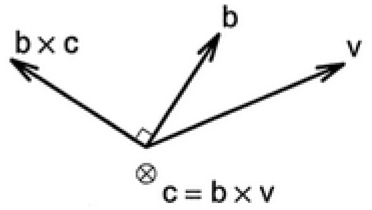

Figure S1.23

The picture shows the plane of **b** and **v**. Because $\mathbf{c} = \mathbf{b} \times \mathbf{v}$, **c** is perpendicular to this plane, as indicated. Therefore $\mathbf{b} \times \mathbf{c}$ lies in the plane and is perpendicular to **b**. This means that **v** can be expressed in terms of the two mutually perpendicular vectors **b** and $\mathbf{b} \times \mathbf{c}$. $\mathbf{v} = \alpha\,\mathbf{b} + \beta\,\mathbf{b} \times \mathbf{c}$ If we form the dot product of this equation with **b** we find that $\mathbf{b} \cdot \mathbf{v} = \alpha \mathbf{b} \cdot \mathbf{b}$. Therefore $\alpha = \mathbf{b} \cdot \mathbf{v}/b^2 = \lambda/b^2$. Similarly, if we form the cross product of the same equation with **b**, we find that $\mathbf{b} \times \mathbf{v} = \beta \mathbf{b} \times (\mathbf{b} \times \mathbf{c})$. Since $\mathbf{b} \times \mathbf{v} = \mathbf{c}$ and (as you can easily check) $\mathbf{b} \times (\mathbf{b} \times \mathbf{c}) = -b^2\mathbf{c}$, this last implies that $\beta = -1/b^2$, and we conclude that $\mathbf{v} = (\lambda\mathbf{b} - \mathbf{b} \times \mathbf{c})/b^2$.

SECTION 1.4 Newton's First and Second Laws; Inertial Frames

1.24 ⋆ In case you haven't studied any differential equations before, I shall be introducing the necessary ideas as needed. Here is a simple exercise to get you started: Find the general solution of the first-order equation $df/dt = f$ for an unknown function $f(t)$. [There are several ways to do this. One is to rewrite the equation as $df/f = dt$ and then integrate both sides.] How many arbitrary constants does the general solution contain? [Your answer should illustrate the important general theorem that the solution to any nth-order differential equation (in a very large class of "reasonable" equations) contains n arbitrary constants.]

1.25 ⋆ Answer the same questions as in Problem 1.24, but for the differential equation $df/dt = -3f$.

Integrating the equation $df/f = -3dt$, we find that $\ln f = -3t + k$, or $f = Ae^{-3t}$, with one arbitrary constant, $A = e^k$.

1.27 ⋆⋆ The hallmark of an inertial reference frame is that any object which is subject to zero net force will travel in a straight line at constant speed. To illustrate this, consider the following experiment: I am standing on the ground (which we shall take to be an inertial frame) beside a perfectly flat horizontal turntable, rotating with constant angular velocity ω. I lean over and shove a frictionless puck so that it slides across the turntable, straight through the center. The puck is subject to zero net force and, as seen from my inertial frame, travels in a straight line. Describe the puck's path as observed by someone sitting at rest on the turntable. This requires careful thought, but you should be able to get a qualitative picture. For a quantitative picture, it helps to use polar coordinates; see Problem 1.46.

Since the puck is frictionless, the net force on it is zero, and, as seen from the ground, it travels in a straight line through the center O, as shown in the left picture. It starts from the point A at $t = 0$, travels "due west" with constant speed v_o, and falls onto the ground at point C after a time $T = 2R/v_o$ (where R is the radius of the turntable).

Now imagine an observer sitting on the turntable near A. As seen from the ground, he is traveling north with speed ωR. Therefore, as seen by the observer, the puck's initial velocity has a sideways (southerly) component ωR, in addition to the westerly component v_o; that is, the puck moves initially west *and* south, as shown in the right picture. (The magnitude of the southerly component depends on the table's rate of rotation ω.) As the puck moves in to a smaller radius r, the sideways component ωr gets less, so the puck's path curves to the right. Continuing to curve, its passes through O and eventually reaches the edge of the turntable at point B. The left picture shows the point B of the table at time $t = 0$. The position of B is determined by the following consideration: In the time $T = 2R/v_o$ for the puck to cross the table, point B of the table must move around to point C where we know the puck falls to the ground. Thus the angle BOC is equal to ωT. The faster the table rotates, the larger the angle BOC and the more sharply the puck's path (as seen from the table) is curved.

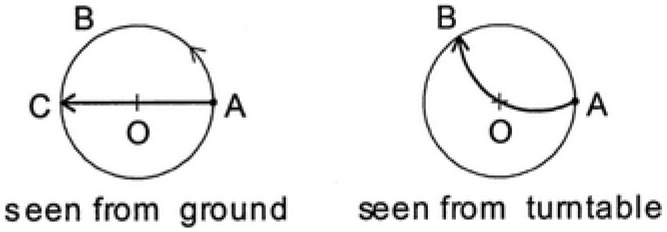

Figure S1.27

SECTION 1.5 The Third Law and Conservation of Momentum

1.28 ⋆ Go over the steps from Equation (1.25) to (1.29) in the proof of conservation of momentum, but treat the case that $N = 3$ and write out all the summations explicitly to be sure you understand the various manipulations.

1.29 ⋆ Do the same tasks as in Problem 1.28 but for the case of four particles ($N = 4$).

When we write out Equations (1.25) and (1.26) for the four particles, we get four equations:

$$\dot{\mathbf{p}}_1 = (\text{net force on particle 1}) = \mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{14} + \mathbf{F}_1^{\text{ext}}$$

$$\dot{\mathbf{p}}_2 = (\text{net force on particle 2}) = \mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_{24} + \mathbf{F}_2^{\text{ext}}$$

$$\dot{\mathbf{p}}_3 = (\text{net force on particle 3}) = \mathbf{F}_{31} + \mathbf{F}_{32} + \mathbf{F}_{34} + \mathbf{F}_3^{\text{ext}}$$

$$\dot{\mathbf{p}}_4 = (\text{net force on particle 4}) = \mathbf{F}_{41} + \mathbf{F}_{42} + \mathbf{F}_{43} + \mathbf{F}_4^{\text{ext}}.$$

Adding these four equations, we find for $\dot{\mathbf{P}} = \dot{\mathbf{p}}_1 + \dot{\mathbf{p}}_2 + \dot{\mathbf{p}}_3 + \dot{\mathbf{p}}_4$,

$$\begin{aligned}\dot{\mathbf{P}} = {} & (\mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{14}) + (\mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_{24}) + (\mathbf{F}_{31} + \mathbf{F}_{32} + \mathbf{F}_{34}) + (\mathbf{F}_{41} + \mathbf{F}_{42} + \mathbf{F}_{43}) \\ & + (\mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}} + \mathbf{F}_3^{\text{ext}} + \mathbf{F}_4^{\text{ext}}).\end{aligned} \tag{S1.1}$$

This corresponds to Equation (1.27). The twelve terms on the first line of the right side can be rearranged to give

$$(\mathbf{F}_{12} + \mathbf{F}_{21}) + (\mathbf{F}_{13} + \mathbf{F}_{31}) + (\mathbf{F}_{14} + \mathbf{F}_{41}) + (\mathbf{F}_{23} + \mathbf{F}_{32}) + (\mathbf{F}_{24} + \mathbf{F}_{42}) + (\mathbf{F}_{34} + \mathbf{F}_{43}) = 0$$

since each of the six pairs is zero by Newton's third law. Thus Equation (S1.1) for $\dot{\mathbf{P}}$ reduces to

$$\dot{\mathbf{P}} = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}} + \mathbf{F}_3^{\text{ext}} + \mathbf{F}_4^{\text{ext}} = \mathbf{F}^{\text{ext}}$$

which is the required Equation (1.29).

1.31 ⋆ In Section 1.5 we proved that Newton's third law implies the conservation of momentum. Prove the converse, that if the law of conservation of momentum applies to every possible group of particles, then the interparticle forces must obey the third law. [*Hint:* However many particles your system contains, you can focus your attention on just two of them. (Call them 1 and 2.) The law of conservation of momentum says that if there are no external forces on this pair of particles, then their total momentum must be constant. Use this to prove that $\mathbf{F}_{12} = -\mathbf{F}_{21}$.]

We have to prove for any pair of particles (call them 1 and 2) that $\mathbf{F}_{12} = -\mathbf{F}_{21}$. To do this, suppose that all forces on 1 and 2, except $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$, have been switched off. For example, we could move all bodies with which 1 and 2 interact to a great distance. If the law of conservation of momentum holds, then $\mathbf{p}_1 + \mathbf{p}_2$ is constant, which implies that $\dot{\mathbf{p}}_1 + \dot{\mathbf{p}}_2 = 0$. Since all other forces have been switched off, the only force on particle 1 is $\mathbf{F}_{12}$ and the only force on 2 is $\mathbf{F}_{21}$. Therefore this last equation implies that $\mathbf{F}_{12} + \mathbf{F}_{21} = 0$, which is the required result. (There is a subtle point here: We have assumed that switching off the external forces did not change the internal forces $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$. One could imagine a world where the presence or absence of external forces affected the internal forces. However, this seems not to be the case in our world.)

1.33 ⋆⋆⋆ If you have some experience in electromagnetism and with vector calculus, prove that the magnetic forces, $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$, between two steady current loops obey Newton's third law. [*Hints:* Let the two currents be I_1 and I_2 and let typical points on the two loops be $\mathbf{r}_1$ and $\mathbf{r}_2$. If $d\mathbf{r}_1$ and $d\mathbf{r}_2$ are short segments of the loops, then according to the Biot–Savart law, the force on $d\mathbf{r}_1$ due to $d\mathbf{r}_2$ is

$$\frac{\mu_o}{4\pi}\frac{I_1 I_2}{s^2}\, d\mathbf{r}_1 \times (d\mathbf{r}_2 \times \hat{\mathbf{s}})$$

where $\mathbf{s} = \mathbf{r}_1 - \mathbf{r}_2$. The force $\mathbf{F}_{12}$ is found by integrating this around both loops. You will need to use the "$BAC - CAB$" rule to simplify the triple product.]

According to the Biot-Savart law, the total force on loop 1 due to loop 2 is

$$\mathbf{F}_{12} = \frac{\mu_o}{4\pi} I_1 I_2 \oint\oint \frac{d\mathbf{r}_1 \times (d\mathbf{r}_2 \times \mathbf{s})}{s^3} = \frac{\mu_o}{4\pi} I_1 I_2 \left\{ \oint\oint d\mathbf{r}_2 \frac{d\mathbf{r}_1 \cdot \mathbf{s}}{s^3} - \oint\oint \frac{\mathbf{s}}{s^3} d\mathbf{r}_1 \cdot d\mathbf{r}_2 \right\}.$$

Here the two integrals run around the two loops, and in writing the second equality I have used the "$BAC = CAB$" rule, that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$. In the first integral on the right, $d\mathbf{r}_1 \cdot \mathbf{s}/s^3 = -d(1/s)$, so when we integrate around loop 1, this gives zero, and we are left with

$$\mathbf{F}_{12} = -\frac{\mu_o}{4\pi} I_1 I_2 \oint\oint \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} d\mathbf{r}_1 \cdot d\mathbf{r}_2.$$

Evidently, $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

SECTION 1.6 Newton's Second Law in Cartesian Coordinates

1.35 ⋆ A golf ball is hit from ground level with speed v_o in a direction that is due east and at an angle θ above the horizontal. Neglecting air resistance, use Newton's second law (1.35) to find the position as a function of time, using coordinates with x measured east, y north, and z vertically up. Find the time for the golf ball to return to the ground and how far it travels in that time.

In the absence of air resistance, the net force on the ball is $\mathbf{F} = m\mathbf{g}$, and with the given choice of axes, $\mathbf{g} = (0, 0, -g)$. Thus Newton's second law, $\mathbf{F} = m\ddot{\mathbf{r}}$, implies that $\ddot{\mathbf{r}} = \mathbf{g}$, or

$$\ddot{x} = 0, \qquad \ddot{y} = 0, \quad \text{and} \quad \ddot{z} = -g.$$

The initial velocity has components $v_{ox} = v_o \cos\theta$, $v_{oy} = 0$, and $v_{oz} = v_o \sin\theta$, and we can choose the initial position to be the origin. The first of the above equations can be integrated once to give $\dot{x} = v_{ox}$, and again to give $x(t) = v_{ox}t$. In the same way, the y equation gives $y(t) = 0$, and the z equation gives $z(t) = v_{oz}t - \frac{1}{2}gt^2$. The ball returns to the ground when $z(t) = 0$ which gives $t = 2v_{oz}/g$. Substituting this time into the expression for $x(t)$ gives the range, range $= 2v_{ox}v_{oz}/g$.

1.37 ⋆ A student kicks a frictionless puck with initial speed v_o, so that it slides straight up a plane that is inclined at an angle θ above the horizontal. **(a)** Write down Newton's second law for the puck and solve to give its position as a function of time. **(b)** How long will the puck take to return to its starting point?

(a) The two forces on the puck are its weight $m\mathbf{g}$ and the normal force $\mathbf{N}$ of the incline. If we choose axes with x measured up the slope, y along the outward normal, and z horizontally across the slope, then $\mathbf{N} = (0, N, 0)$ and $\mathbf{g} = (-g\sin\theta, g\cos\theta, 0)$. Thus Newton's second law reads

$$m\ddot{\mathbf{r}} = \mathbf{N} + m\mathbf{g} \quad \text{or} \quad \begin{cases} m\ddot{x} = -mg\sin\theta \\ m\ddot{y} = N - mg\cos\theta \\ m\ddot{z} = 0 \end{cases}$$

Since $\dot{z} = 0$ initially, it remains so and hence $z = 0$ for all t. The normal force adjusts itself so that $\ddot{y} = 0$, and $y = 0$ for all t. Finally, $\ddot{x} = -g\sin\theta$, which can be integrated twice to give $x = v_o t - \frac{1}{2}gt^2\sin\theta$.

(b) Solving for the times when $x = 0$, we find that $t = 0$ (at launch) or $t = 2v_o/(g\sin\theta)$ (the answer of interest).

1.39 ⋆⋆ A ball is thrown with initial speed v_o up an inclined plane. The plane is inclined at an angle ϕ above the horizontal, and the ball's initial velocity is at an angle θ above the plane. Choose axes with x measured up the slope, y normal to the slope, and z across it. Write down Newton's second law using these axes and find the ball's position as a function of time. Show that the ball lands a distance $R = 2v_o^2\sin\theta\cos(\theta+\phi)/(g\cos^2\phi)$ from its launch point. Show that for given v_o and ϕ, the maximum possible range up the inclined plane is $R_{\max} = v_o^2/[g(1+\sin\phi)]$.

$x = v_o t\cos\theta - \frac{1}{2}gt^2\sin\phi$, $\quad y = v_o t\sin\theta - \frac{1}{2}gt^2\cos\phi$, $\quad z = 0$. When the ball returns to the plane, y is 0, which implies that $t = 2v_o\sin\theta/(g\cos\phi)$. Substituting this time into x and using a couple of trig identities yields the claimed answer for the range R. To find the maximum range, differentiate R with respect to θ and set the derivative equal to zero. This gives $\theta = (\pi - 2\phi)/4$, and substitution into R (plus another trig identity) yields the claimed value of $R_{\max}$.

SECTION 1.7 Two-Dimensional Polar Coordinates

1.41 ⋆ An astronaut in gravity-free space is twirling a mass m on the end of a string of length R in a circle, with constant angular velocity ω. Write down Newton's second law (1.48) in polar coordinates and find the tension in the string.

The r and ϕ components of Newton's second law are $F_r = m(\ddot{r} - r\dot{\phi}^2)$ and $F_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi})$. Since $r = R$ is constant $\dot{r} = \ddot{r} = 0$, and since $\dot{\phi} = \omega$ is constant $\ddot{\phi} = 0$. Finally the only force is the inward tension of the string, so $F_r = -T$. Thus the r equation becomes $-T = -mR\omega^2$, so $T = m\omega^2 R$, which is just the familiar "centripetal force" mv^2/R, since $\omega = v/R$.

1.43 ⋆ **(a)** Prove that the unit vector $\hat{\mathbf{r}}$ of two-dimensional polar coordinates is equal to

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi \tag{1.59}$$

and find a corresponding expression for $\hat{\boldsymbol{\phi}}$. **(b)** Assuming that ϕ depends on the time t, differentiate your answers in part (a) to give an alternative proof of the results (1.42) and (1.46) for the time derivatives $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$.

(a) From Figure 1.11(b), you can see that the x and y components of $\hat{\mathbf{r}}$ are $\cos\phi$ and $\sin\phi$. (Remember that $|\hat{\mathbf{r}}| = 1$.) Therefore, $\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$. Similarly, the x and y components of $\hat{\boldsymbol{\phi}}$ are $-\sin\phi$ and $\cos\phi$, so that $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$.
(b) Differentiating these two results with respect to t and using the chain rule, we find (Remember that $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are constant.)

$$\frac{d}{dt}\hat{\mathbf{r}} = \dot{\phi}(-\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi) = \dot{\phi}\,\hat{\boldsymbol{\phi}} \quad \text{and} \quad \frac{d}{dt}\hat{\boldsymbol{\phi}} = \dot{\phi}(-\hat{\mathbf{x}}\cos\phi - \hat{\mathbf{y}}\sin\phi) = -\dot{\phi}\,\hat{\mathbf{r}}.$$

1.45 ⋆⋆ Prove that if $\mathbf{v}(t)$ is any vector that depends on time (for example the velocity of a moving particle) but which has *constant magnitude*, then $\dot{\mathbf{v}}(t)$ is orthogonal to $\mathbf{v}(t)$. Prove the converse that if $\dot{\mathbf{v}}(t)$ is orthogonal to $\mathbf{v}(t)$, then $|\mathbf{v}(t)|$ is constant. [*Hint:* Consider the derivative of $\mathbf{v}^2$.] This is a very handy result. It explains why, in two-dimensional polars, $d\hat{\mathbf{r}}/dt$ has to be in the direction of $\hat{\boldsymbol{\phi}}$ and vice versa. It also shows that the speed of a charged particle in a magnetic field is constant, since the acceleration is perpendicular to the velocity.

Since the magnitude of $\mathbf{v}(t)$ is the same as $\sqrt{\mathbf{v}(t)\cdot\mathbf{v}(t)}$, the magnitude is constant if and only if $\mathbf{v}(t)\cdot\mathbf{v}(t)$ is. Since

$$\frac{d}{dt}[\mathbf{v}(t)\cdot\mathbf{v}(t)] = 2\mathbf{v}(t)\cdot\dot{\mathbf{v}}(t),$$

this implies that the magnitude of $\mathbf{v}(t)$ is constant if and only if $\mathbf{v}(t)\cdot\dot{\mathbf{v}}(t) = 0$; that is, $\mathbf{v}(t)$ is orthogonal to $\dot{\mathbf{v}}(t)$

1.47 ⋆⋆ Let the position of a point P in three dimensions be given by the vector $\mathbf{r} = (x, y, z)$ in rectangular (or Cartesian) coordinates. The same position can be specified by **cylindrical polar coordinates**, ρ, ϕ, z, which are defined as follows: Let P' denote the projection of P onto the xy plane; that is, P' has Cartesian coordinates $(x, y, 0)$. Then ρ and ϕ are defined as the two-dimensional polar coordinates of P' in the xy plane, while z is the third Cartesian coordinate, unchanged. **(a)** Make a sketch to illustrate the three cylindrical coordinates. Give expressions for ρ, ϕ, z in terms of the Cartesian coordinates x, y, z. Explain in words what ρ is ("ρ is the distance of P from ______"). There are many variants in notation. For instance, some people use r instead of ρ. Explain why this use of r is unfortunate. **(b)** Describe the three unit vectors $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$, $\hat{\mathbf{z}}$ and write the expansion of the position vector $\mathbf{r}$ in terms of these unit vectors. **(c)** Differentiate your last answer twice to find the cylindrical components of the acceleration $\mathbf{a} = \ddot{\mathbf{r}}$ of the particle. To do this, you will need to know the time derivatives of $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$. You could get these from the corresponding two-dimensional results (1.42) and (1.46), or you could derive them directly as in Problem 1.48.

(a) $\rho = \sqrt{x^2 + y^2}$, $\phi = \arctan(y/x)$ (chosen to lie in the right quadrant), and z is the same as in Cartesians. The coordinate ρ is the perpendicular distance from P to the z axis. If we use r for the coordinate ρ, then r is not the same thing as $|\mathbf{r}|$ and $\hat{\mathbf{r}}$ is not the unit vector in the direction of $\mathbf{r}$ [see part (b)].

(b) The unit vector $\hat{\boldsymbol{\rho}}$ points in the direction of increasing ρ (with ϕ and z fixed), that is, directly away from the z axis; $\hat{\boldsymbol{\phi}}$ is tangent to a horizontal circle through P centered on the z axis (counterclockwise, seen from above); $\hat{\mathbf{z}}$ is parallel to the z axis.

$$\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}.$$

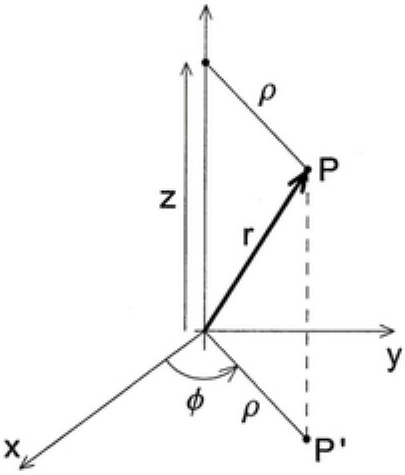

Figure S1.47

(c) Differentiating this equation we find (Remember that $\hat{\mathbf{z}}$ is constant.)

$$\dot{\mathbf{r}} = \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\frac{d\hat{\boldsymbol{\rho}}}{dt} + \dot{z}\hat{\mathbf{z}} = \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\mathbf{z}}$$

since $d\hat{\boldsymbol{\rho}}/dt = \dot{\phi}\hat{\boldsymbol{\phi}}$ [see (1.42)]. Differentiating again, we find similarly that

$$\ddot{\mathbf{r}} = (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\boldsymbol{\rho}} + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\hat{\boldsymbol{\phi}} + \ddot{z}\hat{\mathbf{z}}.$$

1.49 ⋆⋆ Imagine two concentric cylinders, centered on the vertical z axis, with radii $R \pm \epsilon$, where ϵ is very small. A small frictionless puck of thickness 2ϵ is inserted between the two cylinders, so that it can be considered a point mass that can move freely at a fixed distance from the vertical axis. If we use cylindrical polar coordinates (ρ, ϕ, z) for its position (Problem 1.47), then ρ is fixed at $\rho = R$, while ϕ and z can vary at will. Write down and solve Newton's second law for the general motion of the puck, including the effects of gravity. Describe the puck's motion.

There are two forces on the puck, the net normal force of the two cylinders and the force of gravity. So $\mathbf{F} = N\hat{\boldsymbol{\rho}} - mg\hat{\mathbf{z}}$. Since the puck is confined between the cylinders, $\rho = R$, a constant. The three components of $\mathbf{F} = m\mathbf{a}$ are:

$$\begin{aligned} F_\rho &= m(\ddot{\rho} - \rho\dot{\phi}^2) \quad &&\text{or} \quad N = -mR\dot{\phi}^2 \\ F_\phi &= m(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}) \quad &&\text{or} \quad 0 = mR\ddot{\phi} \\ F_z &= m\ddot{z} \quad &&\text{or} \quad -mg = m\ddot{z}. \end{aligned}$$

The ρ equation tells us the magnitude and direction (inward) of the normal force. The ϕ equation tells us that $\dot{\phi}$ is constant. (This is actually conservation of angular momentum.) Thus $\dot{\phi} = \omega$, a constant, and hence $\phi = \phi_o + \omega t$. The puck moves around the cylinder at a constant rate ω. The z equation tells us that $\dot{z} = v_{oz} - gt$ and hence that $z = z_o + v_{oz}t - \frac{1}{2}gt^2$. That is, the vertical motion is precisely that of a body in free fall. The resulting path is a helix of downward increasing pitch.

(b) The unit vector $\hat{\boldsymbol{\rho}}$ points in the direction of increasing ρ (with ϕ and z fixed), that is, directly away from the z axis; $\hat{\boldsymbol{\phi}}$ is tangent to a horizontal circle through P centered on the z axis (counterclockwise, seen from above); $\hat{\mathbf{z}}$ is parallel to the z axis.

$$\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}.$$

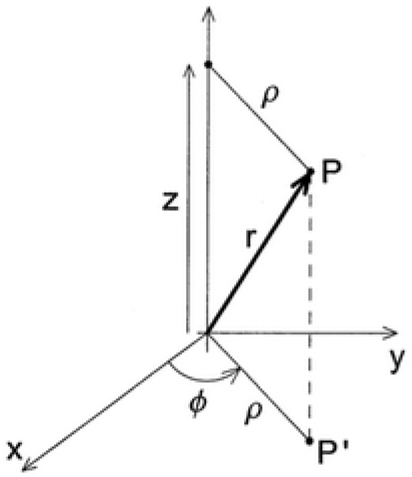

Figure S1.47

(c) Differentiating this equation we find (Remember that $\hat{\mathbf{z}}$ is constant.)

$$\dot{\mathbf{r}} = \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\frac{d\hat{\boldsymbol{\rho}}}{dt} + \dot{z}\hat{\mathbf{z}} = \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\mathbf{z}}$$

since $d\hat{\boldsymbol{\rho}}/dt = \dot{\phi}\hat{\boldsymbol{\phi}}$ [see (1.42)]. Differentiating again, we find similarly that

$$\ddot{\mathbf{r}} = (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\boldsymbol{\rho}} + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\hat{\boldsymbol{\phi}} + \ddot{z}\hat{\mathbf{z}}.$$

1.49 ⋆⋆ Imagine two concentric cylinders, centered on the vertical z axis, with radii $R \pm \epsilon$, where ϵ is very small. A small frictionless puck of thickness 2ϵ is inserted between the two cylinders, so that it can be considered a point mass that can move freely at a fixed distance from the vertical axis. If we use cylindrical polar coordinates (ρ, ϕ, z) for its position (Problem 1.47), then ρ is fixed at $\rho = R$, while ϕ and z can vary at will. Write down and solve Newton's second law for the general motion of the puck, including the effects of gravity. Describe the puck's motion.

There are two forces on the puck, the net normal force of the two cylinders and the force of gravity. So $\mathbf{F} = N\hat{\boldsymbol{\rho}} - mg\hat{\mathbf{z}}$. Since the puck is confined between the cylinders, $\rho = R$, a constant. The three components of $\mathbf{F} = m\mathbf{a}$ are:

$$\begin{aligned} F_\rho &= m(\ddot{\rho} - \rho\dot{\phi}^2) &\quad \text{or} \quad & N = -mR\dot{\phi}^2 \\ F_\phi &= m(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}) &\quad \text{or} \quad & 0 = mR\ddot{\phi} \\ F_z &= m\ddot{z} &\quad \text{or} \quad & -mg = m\ddot{z}. \end{aligned}$$

The ρ equation tells us the magnitude and direction (inward) of the normal force. The ϕ equation tells us that $\dot{\phi}$ is constant. (This is actually conservation of angular momentum.) Thus $\dot{\phi} = \omega$, a constant, and hence $\phi = \phi_o + \omega t$. The puck moves around the cylinder at a constant rate ω. The z equation tells us that $\dot{z} = v_{oz} - gt$ and hence that $z = z_o + v_{oz}t - \frac{1}{2}gt^2$. That is, the vertical motion is precisely that of a body in free fall. The resulting path is a helix of downward increasing pitch.

1.50 ⋆⋆⋆ [Computer] The differential equation (1.51) for the skateboard of Example 1.2 cannot be solved in terms of elementary functions, but is easily solved numerically. **(a)** If you have access to software, such as Mathematica, Maple, or Matlab, that can solve differential equations numerically, solve the differential equation for the case that the board is released from $\phi_o = 20$ degrees, using the values $R = 5$ m and $g = 9.8$ m/s^2. Make a plot of ϕ against time for two or three periods. **(b)** On the same picture, plot the approximate solution (1.57) with the same $\phi_o = 20°$. Comment on your two graphs. Note: If you haven't used the numerical solver before, you will need to learn the necessary syntax. For example, in Mathematica you will need to learn the syntax for "NDSolve" and how to plot the solution that it provides. This takes a bit of time, but is something that is very well worth learning.

1.51 ⋆⋆⋆ [Computer] Repeat all of Problem 1.50 but using the initial value $\phi_o = \pi/2$.

In the picture, the solid curve is a numerical solution of the differential equation, found with Mathematica's NDSolve. The dashed curve is the small-oscillation approximation (1.57) with the same initial condition ($\phi_o = \pi/2$). Considering how large the initial angle is, the small-angle approximation does remarkably well. The only obvious discrepancy is that the approximation oscillates somewhat too fast, as one would expect. (For large amplitudes, the true period is a little longer.) After two complete cycles, the approximate solution is nearly half a cycle ahead. Also, if you look closely you can maybe see that the actual motion is not perfectly sinusoidal at the extremes—the crests are a little wider and flatter.

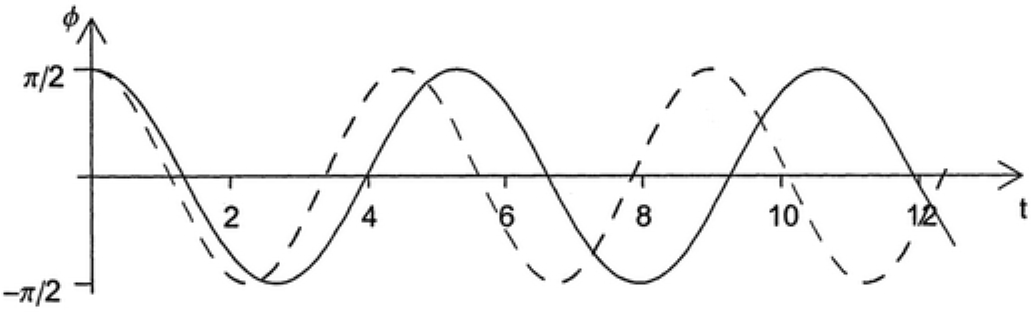

Figure S1.51

CHAPTER 2

Projectiles and Charged Particles

PROBLEMS AND SOLUTIONS

SECTION 2.1 Air Resistance

2.1 ⋆ When a baseball flies through the air, the ratio $f_{\text{quad}}/f_{\text{lin}}$ of the quadratic to the linear drag force is given by (2.7). Given that a baseball has diameter 7 cm, find the approximate speed v at which the two drag forces are equally important. For what approximate range of speeds is it safe to treat the drag force as purely quadratic? Under normal conditions is it a good approximation to ignore the linear term? Answer the same questions for a beach ball of diameter 70 cm.

According to (2.7), $f_{\text{quad}}/f_{\text{lin}} = vD\gamma/\beta$, so this equals 1 when $v = 1/(D\gamma/\beta)$, which is roughly 1 cm/s if $D = 7$ cm. Since baseballs generally travel much faster than 1 cm/s, it is usually a very good approximation to ignore f_{lin}. For a beachball with $D = 70$ cm, the corresponding speed is about 1 mm/s.

2.3 ⋆ **(a)** The quadratic and linear drag forces on a moving sphere in a fluid are given by (2.84) and (2.82) (Problems 2.4 and 2.2). Show that the ratio of these two kinds of drag force can be written as $f_{\text{quad}}/f_{\text{lin}} = R/48$,[1] where the dimensionless **Reynolds number** R is

$$R = \frac{Dv\varrho}{\eta} \tag{2.83}$$

where D is the sphere's diameter, v its speed, and ϱ and η are the fluid's density and viscosity. Clearly the Reynolds number is a measure of the relative importance of the two kinds of drag.[2] When R is very large, the quadratic drag is dominant and the linear can be neglected; vice versa when R is very small. **(b)** Find the Reynolds number for a steel ball bearing (diameter 2 mm) moving at 5 cm/s through glycerin (density 1.3 g/cm^3 and viscosity 12 N·s/m^2 at STP).

(a) From (2.84) and (2.82), $f_{\text{quad}}/f_{\text{lin}} = (\kappa\varrho A v^2)/(3\pi\eta D v)$. With $\kappa = 1/4$ and $A = \pi D^2/4$, this becomes $\varrho D v/(48\eta)$ or $R/48$, with R given by (2.83).
(b) With the given numbers, $R = 1.1 \times 10^{-2}$ and it is very safe to neglect the quadratic drag.

SECTION 2.2 Linear Air Resistance

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[1] The numerical factor 48 is for a sphere. A similar result holds for other bodies, but the numerical factor is different for different shapes.

[2] The Reynolds number is usually defined by (2.83) for flow involving any object, with D defined as a typical linear dimension. One sometimes hears the claim that R is the ratio $f_{\text{quad}}/f_{\text{lin}}$. Since $f_{\text{quad}}/f_{\text{lin}} = R/48$ for a sphere, this claim would be better phrased as "R is roughly of the order of $f_{\text{quad}}/f_{\text{lin}}$."

2.5★ Suppose that a projectile which is subject to a linear resistive force is thrown vertically down with a speed v_{yo} which is *greater* than the terminal speed v_{ter}. Describe and explain how the velocity varies with time, and make a plot of v_y against t for the case that $v_{yo} = 2v_{\text{ter}}$.

With $v_y > v_{\text{ter}}$, the drag force is greater than the weight, and the net force is upward. Thus the projectile slows down, with v_y approaching v_{ter} as $t \to \infty$. This is clear from Eq. (2.30), as shown in the plot.

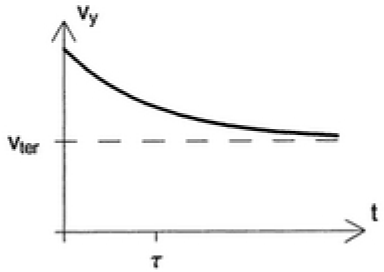


Figure S2.5

2.7★ There are certain simple one-dimensional problems where the equation of motion (Newton's second law) can always be solved, or at least reduced to the problem of doing an integral. One of these (which we have met a couple of times in this chapter) is the motion of a one-dimensional particle subject to a force that depends only on the velocity v, that is, $F = F(v)$. Write down Newton's second law and separate the variables by rewriting it as $m\, dv/F(v) = dt$. Now integrate both sides of this equation and show that

$$t = m \int_{v_o}^{v} \frac{dv'}{F(v')}.$$

Provided you can do the integral, this gives t as a function of v. You can then solve to give v as a function of t. Use this method to solve the special case that $F(v) = F_o$, a constant, and comment on your result. This method of *separation of variables* is used again in Problems 2.8 and 2.9.

With $F = F(v)$, Newton's second law, $F = m\, dv/dt$, can be rewritten as $dt = m\, dv/F(v)$, which can be integrated to give the advertised result. If $F = F_o$, a constant, the integral gives $t = (m/F_o)(v - v_o)$ or $v = v_o + at$, where $a = F_o/m$. This is the well known kinematic formula for v when a is constant.

2.9★ We solved the differential equation (2.29), $m\dot{v}_y = -b(v_y - v_{\text{ter}})$, for the velocity of an object falling through air, by inspection — a most respectable way of solving differential equations. Nevertheless, one would sometimes like a more systematic method, and here is one. Rewrite the equation in the "separated" form

$$\frac{m\, dv_y}{v_y - v_{\text{ter}}} = -b\, dt$$

and integrate both sides from time 0 to t to find v_y as a function of t. Compare with (2.30).

If we integrate the given equation, with the velocity running from v_{yo} to v_y and time from 0 to t, we find $m \ln[(v_y - v_{\text{ter}})/(v_{yo} - v_{\text{ter}})] = -bt$, or, solving for v_y,

$$v_y = v_{\text{ter}} + (v_{y\text{o}} - v_{\text{ter}})e^{-bt/m},$$

which is exactly (2.30), since $m/b = \tau$.

2.11 ⋆⋆ Consider an object that is thrown vertically up with initial speed v_o in a linear medium. **(a)** Measuring y *upward* from the point of release, write expressions for the object's velocity $v_y(t)$ and position $y(t)$. **(b)** Find the time for the object to reach its highest point and its position y_max at that point. **(c)** Show that as the drag coefficient approaches zero, your last answer reduces to the well-known result $y_\text{max} = \frac{1}{2}v_\text{o}^2/g$ for an object in the vacuum. [*Hint:* If the drag force is very small, the terminal speed is very big, so v_o/v_ter is very small. Use the Taylor series for the log function to approximate $\ln(1+\delta)$ by $\delta - \frac{1}{2}\delta^2$. (For a little more on Taylor series see Problem 2.18.)]

(a) Since we are now measuring y upward, the answers can be found from (2.30) and (2.35) by replacing v_ter with $-v_\text{ter}$:

$$v_y(t) = -v_\text{ter} + (v_\text{o} + v_\text{ter})e^{-t/\tau} \quad \text{and} \quad y(t) = -v_\text{ter}t + (v_\text{o} + v_\text{ter})\tau(1 - e^{-t/\tau}).$$

(b) Setting $v_y = 0$ and solving for t, we find $t_\text{top} = \tau\ln(1 + v_\text{o}/v_\text{ter})$. Sustituting this time into $y(t)$ we find $y_\text{max} = [v_\text{o} - v_\text{ter}\ln(1 + v_\text{o}/v_\text{ter})]\tau$.
(c) In the vacuum $v_\text{ter} = \infty$. Letting $v_\text{ter} \to \infty$ in y_max and using the suggested approximation for the log term, we find

$$y_\text{max} \to \left\{v_\text{o} - v_\text{ter}\left[\frac{v_\text{o}}{v_\text{ter}} - \frac{1}{2}\left(\frac{v_\text{o}}{v_\text{ter}}\right)^2\right]\right\}\tau = \frac{v_\text{o}^2}{2g}$$

since the first two terms in the middle expression cancel each other and $v_\text{ter} = g\tau$.

2.12 ⋆⋆ Problem 2.7 is about a class of one-dimensional problems that can always be reduced to doing an integral. Here is another. Show that if the net force on a one-dimensional particle depends only on position, $F = F(x)$, then Newton's second law can be solved to find v as a function of x given by

$$v^2 = v_\text{o}^2 + \frac{2}{m}\int_{x_\text{o}}^{x} F(x')\,dx'. \tag{2.85}$$

[*Hint:* Use the chain rule to prove the following handy relation, which we could call the "$v\,dv/dx$ rule": If you regard v as a function of x, then

$$\dot{v} = v\frac{dv}{dx} = \frac{1}{2}\frac{dv^2}{dx}. \tag{2.86}$$

Use this to rewrite Newton's second law in the separated form $m\,d(v^2) = 2F(x)\,dx$ and then integrate from x_o to x.] Comment on your result for the case that $F(x)$ is actually a constant. (You may recognize your solution as a statement about kinetic energy and work, both of which we shall be discussing in Chapter 4.)

2.13 ⋆⋆ Consider a mass m constrained to move on the x axis and subject to a net force $F = -kx$ where k is a positive constant. The mass is released from rest at $x = x_\text{o}$ at time $t = 0$. Use the result (2.85) in Problem 2.12 to find the mass's speed as a function of x; that is, $dx/dt = g(x)$ for some function $g(x)$. Separate this as $dx/g(x) = dt$ and integrate from time 0 to t to find x as a function of t. (You may recognize this as one way — not the easiest — to solve the simple harmonic oscillator.)

With $F = -kx$ and $v_o = 0$, Eq. (2.85) becomes

$$v^2 = -\frac{2k}{m}\int_{x_o}^{x} x'dx' = \omega^2(x_o^2 - x^2) \quad \text{or} \quad v = -\omega\sqrt{x_o^2 - x^2} \tag{S2.1}$$

where I have introduced the shorthand $\omega^2 = k/m$. [The second result is the square root of the first. Getting the right sign for the square root takes a little thought. Initially the velocity is clearly negative, and this is the phase of the motion I shall consider. After a while, the sign of v changes and the minus sign in (S2.1) must be changed to a plus. Quite surprisingly, the final result is the same either way.] Writing $v = dx/dt$ in (S2.1), rearranging, and integrating, we find that

$$\omega t = -\int_{x_o}^{x} dx'/\sqrt{x_o^2 - x^2} = \arccos(x/x_o) \quad \text{or} \quad x = x_o\cos(\omega t),$$

which is simple harmonic motion. (To do the integral, I used the substitutions $x/x_o = u$ and then $u = \cos\theta$.)

SECTION 2.3 Trajectory and Range in a Linear Medium

2.15 ⋆ Consider a projectile launched with velocity (v_{xo}, v_{yo}) from horizontal ground (with x measured horizontally and y vertically up). Assuming no air resistance, find how long the projectile is in the air and show that the distance it travels before landing (the horizontal range) is $2v_{xo}v_{yo}/g$.

Since the only force is the projectile's weight $m\mathbf{g}$, Newton's second law implies that $\ddot{\mathbf{r}} = \mathbf{g}$ and its two components can be integrated twice to give the well-known results $x = v_{xo}t$ and $y = v_{yo}t - \frac{1}{2}gt^2$ (if we take $x_o = y_o = 0$). At landing, $y = 0$, which gives the time of flight as $t = 2v_{yo}/g$. The range is just the value of x at this time, namely $x = v_{xo}t = 2v_{xo}v_{yo}/g$.

2.17 ⋆ The two equations (2.36) give a projectile's position (x, y) as a function of t. Eliminate t to give y as a function of x. Verify Equation (2.37).

From the first of Equations (2.36) we find that $1 - e^{-t/\tau} = x(t)/(v_{xo}\tau)$ and hence $t = -\tau\ln(1 - x/v_{xo}\tau)$. Substituting these into the second of (2.36), we find

$$y = \frac{v_{yo} + v_{\text{ter}}}{v_{xo}}x + v_{\text{ter}}\tau\ln\left(1 - \frac{x}{v_{xo}\tau}\right)$$

which is (2.37).

2.19 ⋆ Consider the projectile of Section 2.3. **(a)** Assuming there is no air resistance, write down the position (x, y) as a function of t, and eliminate t to give the trajectory y as a function of x. **(b)** The correct trajectory, including a linear drag force, is given by (2.37). Show that this reduces to your answer for part (a) when air resistance is switched off (τ and $v_{\text{ter}} = g\tau$ both approach infinity). [*Hint:* Remember the Taylor series (2.40) for $\ln(1 - \epsilon)$.]

(a) In the absence of air resistance, we know that $x = v_{xo}t$ and $y = v_{yo}t - \frac{1}{2}gt^2$. If we solve the first of these to give $t = x/v_{xo}$ and then substitute into the second, we find

$$y = \frac{v_{yo}}{v_{xo}}x - \tfrac{1}{2}g\left(\frac{x}{v_{xo}}\right)^2,$$

which is the equation of a parabola.
(b) As air resistance is switched off, $\tau \to \infty$, and the second term inside the log term of (2.37) becomes small. Thus we can use the Taylor series (2.40) for the log,

$$\ln\left(1 - \frac{x}{v_{xo}\tau}\right) = -\frac{x}{v_{xo}\tau} - \frac{1}{2}\left(\frac{x}{v_{xo}\tau}\right)^2 - \cdots,$$

in (2.37). For τ sufficiently large, we can neglect all remaining terms in this series and (2.37) becomes

$$y \approx \frac{v_{yo} + v_{\text{ter}}}{v_{xo}}x - v_{\text{ter}}\tau\left(\frac{x}{v_{xo}\tau} + \frac{1}{2}\frac{x^2}{v_{xo}^{\ 2}\tau^2}\right).$$

The second and third terms on the right cancel, and, if we replace v_{ter} by $g\tau$, the two remaining terms give precisely the answer to part (a).

2.21 ★★★ A gun can fire shells in any direction with the same speed v_o. Ignoring air resistance and using cylindrical polar coordinates with the gun at the origin and z measured vertically up, show that the gun can hit any object inside the surface

$$z = \frac{v_o^2}{2g} - \frac{g}{2v_o^2}\rho^2.$$

Describe this surface and comment on its dimensions.

The problem is axially symmetric about the z axis, so we can focus on any one direction ϕ. To be definite, let's take $\phi = 0$. If the gun is fired at an angle θ above the horizontal, then its trajectory is

$$z = v_o t\sin\theta - \tfrac{1}{2}gt^2 \quad \text{and} \quad \rho = v_o t\cos\theta.$$

Solving the second equation for t and substituting into the first, we get the equation of the trajectory:

$$z = \rho\tan\theta - \frac{g\rho^2}{2v_o^2}\sec^2\theta.$$

This is the solid curve in the figure. For any distance ρ from the gun, the highest the shell can go, $z_{\max}$, is found by differentiating z with respect to θ and finding where $\partial z/\partial\theta = 0$. This gives $\tan\theta = v_o^2/g\rho$. (You can easily check that $\partial^2 z/\partial\theta^2 < 0$, so this is a maximum of z.) Substituting into the expression for z, we find

$$z_{\max}(\rho) = \frac{v_o^2}{2g} - \frac{g}{2v_o^2}\rho^2.$$

This is the highest the shell can reach at the distance ρ. The locus of $z_{\max}$ is the dashed parabola shown. By changing the angle θ we can get the shell to pass through any point lower than $z_{\max}$; therefore, all points below the dashed curve are accessible to the gun. When the parabola is rotated about the z axis we get a paraboloid of revolution. Its vertex is at $z = v_o^2/2g$ (well known as the maximum height of a projectile launched vertically up) and its radius in the plane $z = 0$ is $\rho = v_o^2/g$ (well known to be the maximum horizontal range of a gun with muzzle speed v_o).

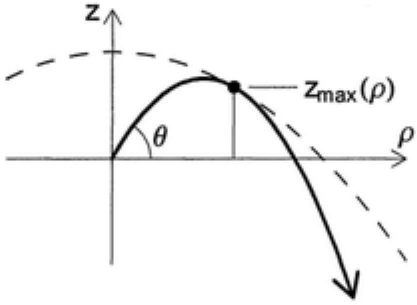


Figure S2.21

SECTION 2.4 Quadratic Air Resistance

2.23 ⋆ Find the terminal speeds in air of **(a)** a steel ball bearing of diameter 3 mm, **(b)** a 16-pound steel shot, and **(c)** a 200-pound parachutist in free fall in the fetal position. In all three cases, you can safely assume the drag force is purely quadratic. The density of steel is about 8 g/cm^3 and you can treat the parachutist as a sphere of density 1 g/cm^3.

According to Eq. (2.59), $v_{\text{ter}} = \sqrt{mg/\gamma D^2}$. Since $m = \frac{4}{3}\pi R^3\varrho = \frac{1}{6}\pi D^3\varrho$, we can eliminate either m or D to give

$$v_{\text{ter}} = \sqrt{\frac{\pi D\varrho g}{6\gamma}} = \left(\frac{\pi\varrho}{6m}\right)^{1/3}\sqrt{\frac{mg}{\gamma}}. \tag{S2.2}$$

In all three cases, $g = 9.8$ m/s^2 and $\gamma = 0.25$ kg/m^3.
(a) With $D = 3$ mm and $\varrho = 8$ g/cm^3, the second expression in Eq. (S2.2) gives $v_{\text{ter}} = 22$ m/s.
(b) With $m = 16 \times 0.454 = 7.26$ kg and $\varrho = 8$ g/cm^3, the third expression in Eq. (S2.2) gives $v_{\text{ter}} = 140$ m/s.
(c) With $m = 200 \times 0.454 = 90.8$ kg and $\varrho = 1$ g/cm^3, the third expression in Eq. (S2.2) gives $v_{\text{ter}} = 107$ m/s.

2.25 ⋆ Consider the cyclist of Section 2.4, coasting to a halt under the influence of a quadratic drag force. Derive in detail the results (2.49) and (2.51) for her velocity and position, and verify that the constant $\tau = m/cv_o$ is indeed a time.

The steps of the derivations are already given in almost complete detail in the text between (2.47) and (2.51). Since the drag force is cv^2, the dimensions of c are $[c] = [F/v^2] = MLT^{-2}/(L/T)^2 = ML^{-1}$. Therefore

$$[\tau] = \left[\frac{m}{cv_o}\right] = \frac{M}{(ML^{-1})(L/T)} = T.$$

2.27 ⋆ I kick a puck of mass m up an incline (angle of slope = θ) with initial speed v_o. There is no friction between the puck and the incline, but there is air resistance with magnitude $f(v) = cv^2$. Write down and solve Newton's second law for the puck's velocity as a function of t on the upward journey. How long does the upward journey last?

If we choose our x axis pointing straight up the slope, then for the upward journey the x component of the second law reads

$$m\dot{v} = -cv^2 - mg\sin\theta = -c(v^2 + v_{\text{ter}}^2)$$

where v denotes the x component of the velocity and I have introduced the terminal speed for the puck on the incline, defined so that $v_{\text{ter}}^2 = (mg\sin\theta)/c$. If we write this in the separated form $m\,dv/(v^2 + v_{\text{ter}}^2) = -c\,dt$, we can integrate both sides (the left side from v_o to v and the right from 0 to t) to give

$$\frac{m}{v_{\text{ter}}}[\arctan(v/v_{\text{ter}}) - \arctan(v_o/v_{\text{ter}})] = -ct \tag{S2.3}$$

which can be solved to give

$$v = v_{\text{ter}}\tan\left(\arctan(v_o/v_{\text{ter}}) - cv_{\text{ter}}t/m\right).$$

Putting $v = 0$ in Eq. (S2.3), we find that the time to reach the top is $t = (m/cv_{\text{ter}})\arctan(v_o/v_{\text{ter}})$.

2.29 ⋆ The terminal speed of a 70-kg skydiver in spread-eagle position is around 50 m/s (about 115 mi/h). Find his speed at times $t = 1, 5, 10, 20, 30$ seconds after he jumps from a stationary balloon. Compare with the corresponding speeds if there were no air resistance.

According to Eq. (2.57), the actual speed is $v = v_{\text{ter}}\tanh(gt/v_{\text{ter}})$, whereas the speed in a vacuum would be just $v_{\text{vac}} = gt$. Putting $v_{\text{ter}} = 50$ m/s and $g = 9.8$ m/s^2, we get the following results:

time (s)	0	1	5	10	20	30
actual speed (m/s)	0	9.7	38	48	50	50
speed in vacuum (m/s)	0	9.8	49	98	196	294

2.31 ⋆⋆ A basketball has mass $m = 600$ g and diameter $D = 24$ cm. **(a)** What is its terminal speed? **(b)** If it is dropped from a 30-m tower, how long does it take to hit the ground and how fast is it going when it does so? Compare with the corresponding numbers in a vacuum.

(a) Using (2.59), we find the basketball's terminal speed is

$$v_{\text{ter}} = \sqrt{\frac{mg}{\gamma D^2}} = \sqrt{\frac{(0.60\text{ kg}) \times (9.8\text{ m/s}^2)}{(0.25\text{ N}\cdot\text{s}^2/\text{m}^4) \times (0.24\text{ m})^2}} = 20.2\text{ m/s}.$$

(b) Solving (2.58), we find that

$$t = (v_{\text{ter}}/g)\,\text{arccosh}[\exp(yg/v_{\text{ter}}^2)] = 2.78\text{ s}$$

where y is the height of the tower ($y = 30$ m). This is to be compared with $t = \sqrt{2y/g} = 2.47$ s in vacuum. According to (2.57), the speed at landing is

$$v = v_{\text{ter}}\tanh(gt/v_{\text{ter}}) = 17.7\text{ m/s},$$

less than, though fairly close to, the terminal speed. In a vacuum the corresponding speed is $v = \sqrt{2gy} = 24.2$ m/s.

2.33 ⋆⋆ The hyperbolic functions cosh z and sinh z are defined as follows:

$$\cosh z = \frac{e^z + e^{-z}}{2} \quad \text{and} \quad \sinh z = \frac{e^z - e^{-z}}{2}$$

for any z, real or complex. **(a)** Sketch the behavior of both functions over a suitable range of real values of z. **(b)** Show that $\cosh z = \cos(iz)$. What is the corresponding relation for $\sinh z$? **(c)** What are the derivatives of $\cosh z$ and $\sinh z$? What about their integrals? **(d)** Show that $\cosh^2 z - \sinh^2 z = 1$. **(e)** Show that $\int dx/\sqrt{1+x^2} = \operatorname{arcsinh} x$. [*Hint:* One way to do this is to make the substitution $x = \sinh z$.]

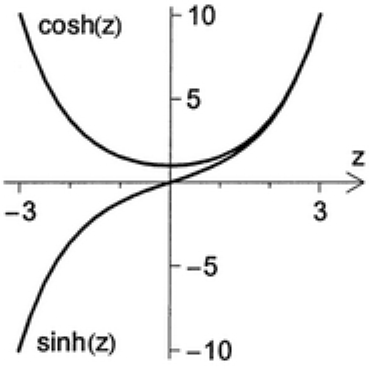


Figure S2.33

(a) Note that when z is large and positive, $\cosh z \approx \sinh z \approx e^z/2$. Similarly, when z is large and negative, $\cosh z \approx -\sinh z \approx e^{-z}/2$. Also $\cosh(0) = 1$ and $\sinh(0) = 0$.

(b) $\cos(iz) = \dfrac{e^{i(iz)} + e^{-i(iz)}}{2} = \dfrac{e^{-z} + e^{z}}{2} = \cosh(z)$. Similarly, $\sinh(z) = -i\,\sin(iz)$.

(c) $\dfrac{d}{dz}\cosh(z) = \dfrac{d}{dz}\dfrac{e^z + e^{-z}}{2} = \dfrac{e^z - e^{-z}}{2} = \sinh(z)$, and likewise $\dfrac{d}{dz}\sinh(z) = \cosh(z)$. Integrating these two results, we find that

$$\int \sinh(z)dz = \cosh(z) \quad \text{and} \quad \int \cosh(z)dz = \sinh(z)$$

(d) $\cosh^2(z) - \sinh^2(z) = [\cos(iz)]^2 - [-i\,\sin(iz)]^2 = [\cos(iz)]^2 + [\sin(iz)]^2 = 1$

(e) If we make the substitution $x = \sinh(z)$, then

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\cosh z\, dz}{\sqrt{1+\sinh^2 z}} = \int dz = z = \operatorname{arcsinh}(x).$$

2.35 ⋆⋆ **(a)** Fill in the details of the arguments leading from the equation of motion (2.52) to Equations (2.57) and (2.58) for the velocity and position of a dropped object subject to quadratic air resistance. Be sure to do the two integrals involved. (The results of Problem 2.34 will help.) **(b)** Tidy the two equations by introducing the parameter $\tau = v_{\text{ter}}/g$. Show that when $t = \tau$, v has reached 76% of its terminal value. What are the corresponding percentages when $t = 2\tau$ and 3τ? **(c)** Show that when $t \gg \tau$, the position is approximately $y \approx v_{\text{ter}}t + \text{const}$. [*Hint:* The definition of $\cosh x$ (Problem 2.33) gives you a simple approximation when x is large.] **(d)** Show that for t small, Equation (2.58) for the position gives $y \approx \frac{1}{2}gt^2$. [Use the Taylor series for $\cosh x$ and for $\ln(1+\delta)$.]

(a) Using (2.53) we can replace c by $c = mg/v_{\text{ter}}^2$ in (2.52) to give (2.54) and thence (2.55)

$$\frac{dv}{1 - (v/v_{\text{ter}})^2} = g\,dt$$

which we can now integrate from time 0 to t to give

$$v_{\text{ter}} \int_0^{v/v_{\text{ter}}} \frac{du}{1-u^2} = gt.$$

(Here I made the substitution $v/v_{\text{ter}} = u$, so that $dv = v_{\text{ter}}\, du$.) The integral on the left can be done several ways. One route is to substitute $u = \tanh w$, so that $du = \text{sech}^2 w dw$ and $1 - u^2 = \text{sech}^2 w$, and the integral becomes $v_{\text{ter}} \int dw$. Thus

$$v_{\text{ter}} \operatorname{arctanh}(v/v_{\text{ter}}) = gt$$

which can be solved at once for v to give (2.57).

To find the position $y(t)$ we have only to integrate (2.57) for $v(t)$. Substitute $u = gt/v_{\text{ter}}$, and all you have to do is integrate $\tanh u$. The simple way to do this is to write $\tanh u = \sinh u/\cosh u$ and then $w = \cosh u$, so that $(\tanh u)\, du = dw/w$. The required integral is now just $\int dw/w$, and you get (2.58) directly.

(b) If $\tau = v_{\text{ter}}/g$, we can rewrite the two equations as

$$v(t) = v_{\text{ter}} \tanh(t/\tau) \quad \text{and} \quad y(t) = v_{\text{ter}}\tau \ln[\cosh(t/\tau)].$$

The first of these gives the following values for $v(t)$

time t	τ	2τ	3τ
velocity $v(t)$	$0.76v_{\text{ter}}$	$0.96v_{\text{ter}}$	$0.995v_{\text{ter}}$
percent	76%	96%	99.5%

(c) If $x \gg 1$, then $\cosh x \approx \frac{1}{2}e^x$. Thus if $t \gg \tau$, then

$$y(t) \approx v_{\text{ter}}\tau \ln\left(\tfrac{1}{2}e^{t/\tau}\right) = v_{\text{ter}}\tau\ (t/\tau - \ln 2) = v_{\text{ter}}t + \text{const.}$$

That is, the ball eventually falls with constant velocity.

(d) If $x \ll 1$, then $\cosh x \approx 1 + \frac{1}{2}x^2$. Thus if $t \ll \tau$, the expression of part (b) for $y(t)$ becomes

$$y(t) \approx v_{\text{ter}}\tau \ln[1 + \tfrac{1}{2}(t/\tau)^2] \approx v_{\text{ter}}\tau[\tfrac{1}{2}(t/\tau)^2] = \tfrac{1}{2}gt^2$$

where in the last step I replaced v_{ter} by $g\tau$. That is, at first, air resistance is negligible, and the ball falls freely.

2.37 ⋆⋆ The result (2.57) for the velocity of a falling object was found by integrating Equation (2.55) and the quickest way to do this is to use the integral $\int du/(1 - u^2) = \operatorname{arctanh} u$. Here is another way to do it: Integrate (2.55) using the method of "partial fractions," writing

$$\frac{1}{1-u^2} = \frac{1}{2}\left(\frac{1}{1+u} + \frac{1}{1-u}\right),$$

which lets you do the integral in terms of natural logs. Solve the resulting equation to give v as a function of t and show that your answer agrees with (2.57).

Starting from (2.55) and substituting $u = v/v_{\text{ter}}$, we get $du/(1 - u^2) = g\, dt/v_{\text{ter}}$. Separating into partial fractions as suggested and integrating from time 0 to t, we find

$$\frac{1}{2}\int_0^u \left(\frac{1}{1+u'} + \frac{1}{1-u'}\right) du' = \frac{gt}{v_{\text{ter}}} = T,$$

say. Thus $\ln(1 + u)/(1 - u) = 2T$ and hence $(1 + u)/(1 - u) = e^{2T}$. Solving for u, we find

$$u = \frac{e^{2T} - 1}{e^{2T} + 1} = \frac{e^T - e^{-T}}{e^T + e^{-T}} = \frac{\sinh T}{\cosh T} = \tanh T.$$

Since $u = v/v_{\text{ter}}$ and $T = gt/v_{\text{ter}}$, this is the same as (2.57).

2.39 ⋆⋆ When a cyclist coasts to a stop, he is actually subject to two forces, the quadratic force of air resistance, $f = -cv^2$ (with c as given in Problem 2.26), and a constant frictional force $f_{\rm fr}$ of about 3 N. The former is dominant at high and medium speeds, the latter at low speed. (The frictional force is a combination of ordinary friction in the bearings and rolling friction of the tires on the road.) **(a)** Write down the equation of motion while the cyclist is coasting to a stop. Solve it by separating variables to give t as a function of v. **(b)** Using the numbers of Problem 2.26 (and the value $f_{\rm fr} = 3$ N given above) find how long it takes the cyclist to slow from his initial 20 m/s to 15 m/s. How long to slow to 10 and 5 m/s? How long to come to a full stop? If you did Problem 2.26, compare with the answers you got there ignoring friction entirely.

(a) The equation of motion is $m\dot{v} = -f_{\rm fr} - cv^2$, which separates to give

$$-m\frac{dv}{f_{\rm fr} + cv^2} = dt.$$

This can be integrated from time 0 to t (and velocity from $v_{\rm o}$ to v). The integral over v gives an arctan function. (Make the substitutions $cv^2/f_{\rm fr} = u^2$ and then $u = \tan w$.) The result is

$$t = \frac{m}{\sqrt{f_{\rm fr}c}}\left(\arctan\sqrt{\frac{c}{f_{\rm fr}}}v_{\rm o} - \arctan\sqrt{\frac{c}{f_{\rm fr}}}v\right).$$

(b) Putting in the numbers, with $v_{\rm o} = 20$ m/s and the four given final velocities $v = 15$, 10, 5, and 0 m/s, we find the following corresponding times:

v (m/s)	15	10	5	0
t (s)	6.3	18.4	48.3	142

The corresponding times if we neglect friction are (from Problem 2.26) 6.7, 20.0, 60.0, and ∞. To neglect friction, compared to the quadratic air resistance, is quite good at higher speeds, but terrible at very low speeds.

2.41 ⋆⋆ A baseball is thrown vertically up with speed $v_{\rm o}$ and is subject to a quadratic drag with magnitude $f(v) = cv^2$. Write down the equation of motion for the upward journey (measuring y vertically *up*) and show that it can be rewritten as $\dot{v} = -g[1 + (v/v_{\rm ter})^2]$. Use the "$v\,dv/dx$ rule" (2.86) to write $\dot{v}$ as $v\,dv/dy$, and then solve the equation of motion by separating variables (put all terms involving v on one side and all terms involving y on the other). Integrate both sides to give y in terms of v, and hence v as a function of y. Show that the baseball's maximum height is

$$y_{\rm max} = \frac{v_{\rm ter}^2}{2g}\ln\left(\frac{v_{\rm ter}^2 + v_{\rm o}^2}{v_{\rm ter}^2}\right). \qquad (2.89)$$

If $v_{\rm o} = 20$ m/s (about 45 mph) and the baseball has the parameters given in Example 2.5 (page 61), what is $y_{\rm max}$? Compare with the value in a vacuum.

On the upward journey the velocity v (measured upward) satisfies $m\dot{v} = -mg - cv^2$. Since the terminal speed satisfies $cv_{\rm ter}^2 = mg$, this can be rewritten as $\dot{v} = -g(1 + v^2/v_{\rm ter}^2)$, or, using the "$vdv/dx$" rule, $d(v^2)/(1 + v^2/v_{\rm ter}^2) = -2\,g\,dy$. Integrating this, from $v_{\rm o}$ to v on the left and 0 to y on the right, we find that

$$v_{\rm ter}^2\,[\ln(v^2 + v_{\rm ter}^2) - \ln(v_{\rm o}^2 + v_{\rm ter}^2)] = -2gy, \qquad (S2.4)$$

whence $v = \sqrt{(v_o^2 + v_{\text{ter}}^2)e^{-2gy/v_{\text{ter}}^2} - v_{\text{ter}}^2}$. Putting $v = 0$ in (S2.4) gives $y_{\max}$ exactly as claimed in Eq. (2.89). With the given numbers this yields $y_{\max} = 17.7$ m, compared with the vacuum value, $y_{\max} = v_o^2/2g = 20.4$ m.

2.43 ⋆⋆⋆ [Computer] The basketball of Problem 2.31 is thrown from a height of 2 m with initial velocity $\mathbf{v}_o = 15$ m/s at 45° above the horizontal. **(a)** Use appropriate software to solve the equations of motion (2.61) for the ball's position (x, y) and plot the trajectory. Show the corresponding trajectory in the absence of air resistance. **(b)** Use your plot to find how far the ball travels in the horizontal direction before it hits the floor. Compare with the corresponding range in a vacuum.

(a) We have to solve the differential equations (2.61),

$$m\ddot{x} = -c\sqrt{\dot{x}^2 + \dot{y}^2}\,\dot{x}$$

$$m\ddot{y} = -mg - c\sqrt{\dot{x}^2 + \dot{y}^2}\,\dot{y}$$

with the initial conditions $x(0) = 0,\ \ y(0) = 2,\ \ \dot{x}(0) = 15\cos(45°)$, and $\dot{y}(0) = 15\sin(45°)$. (I've chosen my origin on the ground directly below the launch point. The constant c is $c = \gamma D^2$, as usual, with $\gamma = 0.25$ and $D = 0.24$ m.) Ignoring air resistance, I estimated that the ball would land after about 2.5 sec, so I solved the equations (using NDSolve in Mathematica) for $0 \le t \le 2.5$ s and plotted the solution as the continuous curve shown. The vacuum solution is $x = \dot{x}(0)t$ and $y = y(0) + \dot{y}(0)t - \frac{1}{2}gt^2$, which I have plotted as the dashed curve.

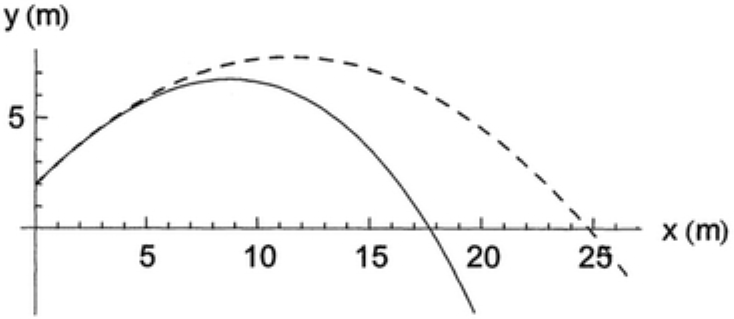


Figure S2.43

(b) From the plots, you can read off the true range as 17.7 m and the range in vacuum as 24.8 m.

SECTION 2.6 Complex Exponentials

2.45 ⋆ **(a)** Using Euler's relation (2.76), prove that any complex number $z = x + iy$ can be written in the form $z = re^{i\theta}$, where r and θ are real. Describe the significance of r and θ with reference to the complex plane. **(b)** Write $z = 3 + 4i$ in the form $z = re^{i\theta}$. **(c)** Write $z = 2e^{-i\pi/3}$ in the form $x + iy$.

(a) Let the polar coordinates of $\mathbf{r} = (x, y)$ be r and θ, so that $x = r\cos\theta$ and $y = r\sin\theta$. Then

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

(b) With $x = 3$ and $y = 4$, $r = 5$ and $\theta = 0.927$ (or about 53°).
(c) $z = 2e^{-i\pi/3} = 1 - i\sqrt{3}$.

2.47 ⋆ For each of the following two pairs of numbers compute $z + w$, $z - w$, zw, and z/w.

$$\textbf{(a)}\ z = 6 + 8i \quad \text{and} \quad w = 3 - 4i \qquad \textbf{(b)}\ z = 8e^{i\pi/3} \quad \text{and} \quad w = 4e^{i\pi/6}.$$

Notice that for adding and subtracting complex numbers, the form $x + iy$ is more convenient, but for multiplying and especially dividing, the form $re^{i\theta}$ is more convenient. In part (a), a clever trick for finding z/w without converting to the form $re^{i\theta}$ is to multiply top and bottom by w^*; try this one both ways.

(a) $z = 6 + 8i = 10e^{i\theta}$ and $w = 3 - 4i = 5e^{-i\theta}$, where $\theta = 0.927$ rad. (Note that the phase angles of z and w are exactly opposite—same θ in both expresions.) Therefore

$$z + w = 9 + 4i \quad \text{and} \quad z - w = 3 + 12i,$$

$$zw = (10e^{i\theta})(5e^{-i\theta}) = 50,$$

and

$$\frac{z}{w} = \frac{10e^{i\theta}}{5e^{-i\theta}} = 2e^{2i\theta} = 2\cos(2\theta) + 2i\sin(2\theta) = -0.56 + 1.92i,$$

or

$$\frac{z}{w} = \frac{zw^*}{ww^*} = \frac{(6+8i)(3+4i)}{(3-4i)(3+4i)} = \frac{-14+48i}{25} = -0.56 + 1.92i.$$

(b) $z = 8e^{i\pi/3} = 4 + 4\sqrt{3}i$ and $w = 4e^{i\pi/6} = 2\sqrt{3} + 2i$. Therefore,

$$z + w = (4 + 2\sqrt{3}) + (4\sqrt{3} + 2)i \quad \text{and} \quad z - w = (4 - 2\sqrt{3}) + (4\sqrt{3} - 2)i,$$

$$zw = (8e^{i\pi/3})(4e^{i\pi/6}) = 32e^{i\pi/2} = 32i \quad \text{and} \quad \frac{z}{w} = \frac{8e^{i\pi/3}}{4e^{i\pi/6}} = 2e^{i\pi/6} = \sqrt{3} + i.$$

2.49 ⋆ Consider the complex number $z = e^{i\theta} = \cos\theta + i\sin\theta$. **(a)** By evaluating z^2 two different ways, prove the trig identities $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$. **(b)** Use the same technique to find corresponding identities for $\cos 3\theta$ and $\sin 3\theta$.

(a) On the one hand, $z^2 = e^{2i\theta} = \cos 2\theta + i\sin 2\theta$. On the other hand, $z^2 = (\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta$. Equating these two expressions, we find

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \quad \text{and} \quad \sin 2\theta = 2\cos\theta\sin\theta.$$

(b) In exactly the same way, we can show that

$$\cos 3\theta = \cos\theta(\cos^2\theta - 3\sin^2\theta) \quad \text{and} \quad \sin 3\theta = \sin\theta(3\cos^2\theta - \sin^2\theta).$$

2.51 ⋆⋆ Use the series definition (2.72) of e^z to prove that $e^z e^w = e^{z+w}$. [*Hint:* If you write down the left side as a product of two series, you will have a huge sum of terms like $z^n w^m$. If you group together all the terms for which $n + m$ is the same (call it p) and use the binomial theorem, you will find you have the series for the right side.]

$$\begin{aligned}
e^z e^w &= \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)\left(1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots\right) \\
&= 1 + (z + w) + \frac{1}{2!}(z^2 + 2zw + w^2) + \frac{1}{3!}(z^3 + 3z^2w + 3zw^2 + w^3) + \cdots \\
&= 1 + (z + w) + \frac{1}{2!}(z + w)^2 + \frac{1}{3!}(z + w)^3 + \cdots \\
&= e^{z+w}.
\end{aligned}$$

SECTION 2.7 Solution for the Charge in a B Field

2.53 ⋆ A charged particle of mass m and positive charge q moves in uniform electric and magnetic fields, **E** and **B**, both pointing in the z direction. The net force on the particle is $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. Write down the equation of motion for the particle and resolve it into its three components. Solve the equations and describe the particle's motion.

The components of the force are $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = q(v_y B, -v_x B, E)$, so the three components of $m\mathbf{a} = \mathbf{F}$ are

$$m\dot{v}_x = qBv_y, \quad m\dot{v}_y = -qBv_x, \quad m\dot{v}_z = qE.$$

The first two of these are exactly the same as (2.64) and (2.65) for the case of no electric field, and the motion of x and y is therefore the same as in Figure 2.15: The transverse position (x, y) moves clockwise around a circle at constant angular velocity $\omega = qB/m$. The equation for v_z shows that there is a constant acceleration in the z direction, $a_z = qE/m$, so that $z = z_o + v_{zo}t + \frac{1}{2}a_z t^2$. The particle moves in a helix or spiral of constant radius around a line parallel to the z axis, with an increasing pitch as the motion in the z direction accelerates.

2.55 ⋆⋆⋆ A charged particle of mass m and positive charge q moves in uniform electric and magnetic fields, **E** pointing in the y direction and **B** in the z direction (an arrangement called "crossed E and B fields"). Suppose the particle is initially at the origin and is given a kick at time $t = 0$ along the x axis with $v_x = v_{xo}$ (positive or negative). **(a)** Write down the equation of motion for the particle and resolve it into its three components. Show that the motion remains in the plane $z = 0$. **(b)** Prove that there is a unique value of v_{xo}, called the drift speed v_{dr}, for which the particle moves undeflected through the fields. (This is the basis of velocity selectors, which select particles traveling at one chosen speed from a beam with many different speeds.) **(c)** Solve the equations of motion to give the particle's velocity as a function of t, for arbitrary values of v_{xo}. [*Hint:* The equations for (v_x, v_y) should look very like Equations (2.68) except for an offset of v_x by a constant. If you make a change of variables of the form $u_x = v_x - v_{\text{dr}}$ and $u_y = v_y$, the equations for (u_x, u_y) will have exactly the form (2.68), whose general solution you know.] **(d)** Integrate the velocity to find the position as a function of t and sketch the trajectory for various values of v_{xo}.

(a) The given fields are $\mathbf{E} = (0, E, 0)$ and $\mathbf{B} = (0, 0, B)$, so the force is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = q(v_y B, E - v_x B, 0).$$

If we define $\omega = qB/m$ as in (2.67), the equation of motion, $\dot{\mathbf{v}} = \mathbf{F}/m$ can be written in components as

$$\dot{v}_x = \omega v_y, \; \dot{v}_y = -\omega(v_x - E/B), \quad \text{and} \quad \dot{v}_z = 0. \tag{S2.5}$$

Since $\dot{v}_z = 0$ and $v_{zo} = 0$ it is clear that $v_z = 0$ for all times, and the motion remains in the plane $z = 0$.

(b) The particle is undeflected if and only if both $\dot{v}_x$ and $\dot{v}_y$ remain zero for all times. From the second of Equations (S2.5) this requires that $v_x = E/B = v_{\text{dr}}$, say. If this condition is satisfied, then v_y remains zero, and the first of Equations (S2.5) implies that v_x remains constant, $v_x = v_{\text{dr}}$. The condition that $v = E/B$ is easily understood: With this velocity, the electric force qE in the $+y$ direction exactly balances the magnetic force qvB in the $-y$ direction.

(c) If we make the suggested change of variables,

$$u_x = v_x - v_{\text{dr}} \quad \text{and} \quad u_y = v_y$$

with $v_{\text{dr}} = E/B$, then the first two of Equations (S2.5) become

$$\dot{u}_x = \omega u_y \quad \text{and} \quad \dot{u}_y = -\omega u_x.$$

These two equations have exactly the form of Equations (2.68), and their solution is therefore $u_x + iu_y = Ae^{i\omega t}$, where, setting $t = 0$, we see that $A = v_{xo} - v_{\text{dr}}$. Rewritten in terms of $\mathbf{v}$, this solution is

$$v_x = v_{\text{dr}} + (v_{xo} - v_{\text{dr}})\cos\omega t \quad \text{and} \quad v_y = -(v_{xo} - v_{\text{dr}})\sin\omega t \tag{S2.6}$$

and, of course, $v_z = 0$. The transverse velocity (v_x, v_y) goes steadily around a circle of radius $(v_{xo} - v_{\text{dr}})$, with a constant drift v_{dr} in the x direction superposed.

(d) To find the position as a function of time, we have only to integrate the equations (S2.6) for the velocity. Before we do this, it is helpful to define the length $R = (v_{xo} - v_{\text{dr}})/\omega$, in terms of which we get

$$x = v_{\text{dr}}t + R\sin\omega t \quad \text{and} \quad y = R(\cos\omega t - 1).$$

The point $(x, y) = R(\sin\omega t, -\cos\omega t)$ describes a circle of radius R about the origin. Thus the charge q describes a circle about the point $(v_{\text{dr}}t, -R)$, which moves steadily to the right with speed v_{dr}. The resulting curve is a cycloid, whose precise appearance depends on the initial velocity v_{xo}, as illustrated below for seven different values of v_{xo}. Notice, in particular, that if $v_{xo} = v_{\text{dr}}$ then $R = 0$ and the charge drifts straight through the fields, as we already knew. (The values of v_{xo} are shown as multiples of v_{dr}.)

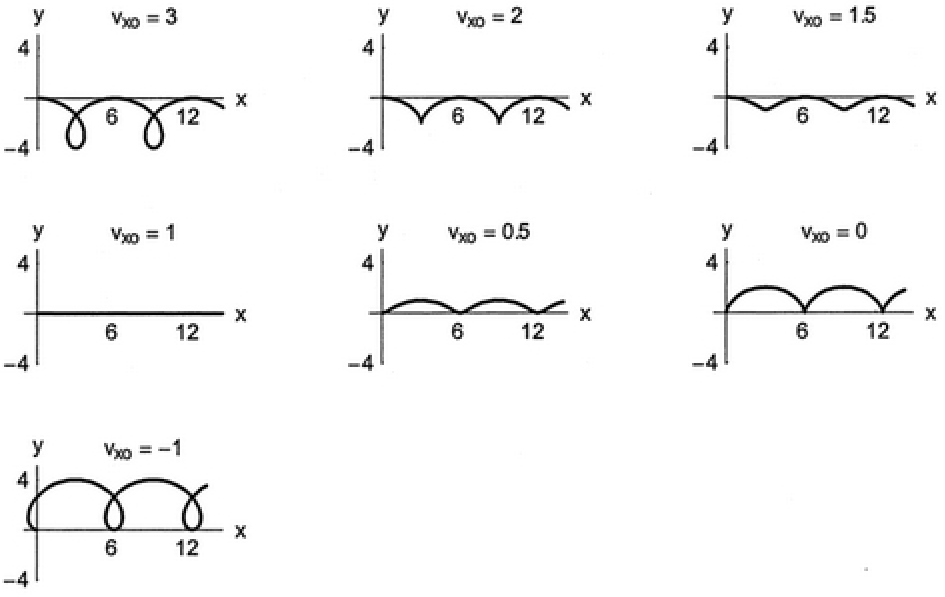


Figure S2.55

CHAPTER 3

Momentum and Angular Momentum

PROBLEMS AND SOLUTIONS

SECTION 3.1 Conservation of Momentum

3.1 ⋆ Consider a gun of mass M (when unloaded) that fires a shell of mass m with muzzle speed v. (That is, the shell's speed relative to the gun is v.) Assuming that the gun is completely free to recoil (no external forces on gun or shell), use conservation of momentum to show that the shell's speed relative to the ground is $v/(1 + m/M)$.

Let v_s and v_g denote the speeds of the shell and gun relative to the ground. Then conservation of momentum implies that $mv_s = Mv_g$. (The shell and gun certainly move in opposite directions.) Their relative speed is $v = v_s + v_g$. Eliminating v_g between these two equations, we find that $v_s = Mv/(m + M)$, which is the desired result.

3.3 ⋆ A shell traveling with velocity $\mathbf{v}_o$ explodes into three pieces of equal masses. Just after the explosion, one piece has velocity $\mathbf{v}_1 = \mathbf{v}_o$ and the other two have velocities $\mathbf{v}_2$ and $\mathbf{v}_3$ that are equal in magnitude ($v_2 = v_3$) but mutually perpendicular. Find $\mathbf{v}_2$ and $\mathbf{v}_3$ and sketch the three velocities.

Let the original mass of the shell be $3m$ (so that each of the fragments has mass m). Conservation of momentum implies that $m\mathbf{v}_1 + m\mathbf{v}_2 + m\mathbf{v}_3 = 3m\mathbf{v}_o$. Since $\mathbf{v}_1 = \mathbf{v}_o$, this simplifies to $\mathbf{v}_2 + \mathbf{v}_3 = 2\mathbf{v}_o$. Squaring both sides and recalling that $\mathbf{v}_2$ is orthogonal to $\mathbf{v}_3$, we find that $v_2^2 + v_3^2 = 4v_o^2$, or, since $v_2 = v_3$, $v_2 = v_3 = \sqrt{2}\,v_o$. Notice that $\mathbf{v}_2$ and $\mathbf{v}_3$ are at 45° on either side of the initial direction.

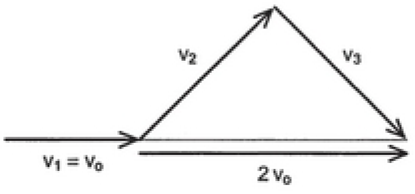


Figure S3.3

3.5 ⋆⋆ Many applications of conservation of momentum involve conservation of energy as well, and we haven't yet begun our discussion of energy. Nevertheless, you know enough about energy from your introductory physics course to handle some problems of this type. Here is one elegant example: An *elastic collision* between two bodies is defined as a collision in which the total kinetic energy of the two bodies after the collision is the same as that before. (A familiar example is the collision between two billiard balls, which generally lose extremely little of their total kinetic energy.) Consider an elastic collision between two equal mass bodies, one of which is initially at rest. Let their velocities be $\mathbf{v}_1$ and $\mathbf{v}_2 = 0$ before the collision, and $\mathbf{v}'_1$ and $\mathbf{v}'_2$ after. Write down the vector equation representing conservation of momentum and the scalar equation which expresses that the collision is elastic. Use these to prove that the angle between $\mathbf{v}'_1$ and $\mathbf{v}'_2$ is 90°. This result was important in the history of atomic and nuclear physics: That two bodies emerged from a collision traveling on perpendicular paths was strongly suggestive that they had equal mass and had undergone an elastic collision.

By conservation of momentum, $m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{v}'_1 + m_2\mathbf{v}'_2$. Since $m_1 = m_2$ and $\mathbf{v}_2 = 0$, this reduces to

$$\mathbf{v}_1 = \mathbf{v}'_1 + \mathbf{v}'_2. \tag{S3.1}$$

Similarly, that the collision is elastic implies that $\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2$, whence

$$v_1^2 = v_1'^2 + v_2'^2 \tag{S3.2}$$

Squaring Equation (S3.1), we find that

$$v_1^2 = v_1'^2 + v_2'^2 + 2\mathbf{v}'_1 \cdot \mathbf{v}'_2,$$

and comparing this with Equation (S3.2), we conclude that $\mathbf{v}'_1 \cdot \mathbf{v}'_2 = 0$. That is $\mathbf{v}'_1$ is perpendicular to $\mathbf{v}'_2$ (except when one of them is zero, in which case the angle between them is undefined).

SECTION 3.2 Rockets

3.7★ The first couple of minutes of the launch of a space shuttle can be described very roughly as follows: The initial mass is 2×10^6 kg, the final mass (after 2 minutes) is about 1×10^6 kg, the average exhaust speed v_{ex} is about 3000 m/s, and the initial velocity is, of course, zero. If all this were taking place in outer space, with negligible gravity, what would be the shuttle's speed at the end of this stage? What is the thrust during the same period and how does it compare with the initial total weight of the shuttle (on earth)?

Since $v_{\text{o}} = 0$, the final velocity as given by (3.8) is

$$v = v_{\text{ex}} \ln(m_{\text{o}}/m) = 3000 \ln(2) \approx 2100 \text{ m/s}.$$

The thrust is given by (3.7) as

$$\text{thrust} = -\dot{m}v_{\text{ex}} = \frac{m_{\text{o}} - m}{t} v_{\text{ex}} \approx 2.5 \times 10^7 \text{ N}.$$

This is just a little bigger than the initial weight, $m_{\text{o}}g \approx 2.0 \times 10^7$ N.

3.9★ From the data in Problem 3.7 you can find the space shuttle's initial mass and the rate of ejecting mass $-\dot{m}$ (which you may assume is constant). What is the minimum exhaust speed v_{ex} for which the shuttle would just begin to lift as soon as burn is fully underway? [*Hint:* The thrust must at least balance the shuttle's weight.]

From the data of Problem 3.7, $m_{\text{o}} = 2 \times 10^6$ kg, and $\dot{m} \approx -8.33 \times 10^3$ kg/s. The minimum exhaust speed is determined by the condition: thrust = $-\dot{m}v_{\text{ex}} = m_{\text{o}}g$, which gives $v_{\text{ex}} = -m_{\text{o}}g/\dot{m} \approx 2350$ m/s, compared with the actual value of about 3000 m/s.

3.11 ⋆⋆ **(a)** Consider a rocket traveling in a straight line subject to an external force F^{ext} acting along the same line. Show that the equation of motion is

$$m\dot{v} = -\dot{m}v_{\text{ex}} + F^{\text{ext}}. \tag{3.29}$$

[Review the derivation of Equation (3.6) but keep the external force term.] **(b)** Specialize to the case of a rocket taking off vertically (from rest) in a gravitational field g, so the equation of motion becomes

$$m\dot{v} = -\dot{m}v_{\text{ex}} - mg. \tag{3.30}$$

Assume that the rocket ejects mass at a constant rate, $\dot{m} = -k$ (where k is a positive constant), so that $m = m_{\text{o}} - kt$. Solve equation (3.30) for v as a function of t, using separation of variables (that is, rewriting the equation so that all terms involving v are on the left and all terms involving t on the right). **(c)** Using the rough data from Problem 3.7, find the space shuttle's speed two minutes into flight, assuming (what is nearly true) that it travels vertically up during this period and that g doesn't change appreciably. Compare with the corresponding result if there were no gravity. **(d)** Describe what would happen to a rocket that was designed so that the first term on the right of Equation (3.30) was smaller than the initial value of the second.

(a) From (3.4) we know that the change in momentum (of rocket plus ejected fuel) in time dt is $dP = m\,dv + dm\,v_{\text{ex}}$. On the other hand we know on general grounds that $\dot{P} = F^{\text{ext}}$, so that $dP = F^{\text{ext}}dt$. Equating these two expressions for dP, we conclude that

$$m\,dv + dm\,v_{\text{ex}} = F^{\text{ext}}dt$$

or, dividing by dt and rearranging,

$$m\dot{v} = -\dot{m}v_{\text{ex}} + F^{\text{ext}}.$$

(b) With $-\dot{m} = k$ (a positive constant) and $F^{\text{ext}} = -mg$, the equation of motion becomes

$$m\dot{v} = kv_{\text{ex}} - mg$$

and since $\dot{m} = -k$ it follows that $m = m_{\text{o}} - kt$. The differential equation separates to give

$$dv = \left(\frac{kv_{\text{ex}}}{m_{\text{o}} - kt} - g\right)dt$$

which can be integrated from time 0 to t (and velocity 0 to v) to give

$$v = v_{\text{ex}}\ln\left(\frac{m_{\text{o}}}{m_{\text{o}} - kt}\right) - gt = v_{\text{ex}}\ln\left(\frac{m_{\text{o}}}{m}\right) - gt.$$

(c) Putting in the numbers ($v_{\text{ex}} = 3000$ m/s, $m_{\text{o}}/m = 2$, $g = 9.8\,\text{m/s}^2$, and $t = 120$ s), we find $v = 900$ m/s. With $g = 0$, the corresponding number is 2100 m/s, so gravity reduces the speed acquired in the first two minutes to a little less than half its weight-free value.

(d) If the thrust kv_{ex} is less than the weight mg, the rocket will just sit on the ground until it has shed enough mass that the thrust *can* overcome the weight. Not a good design!

3.13 ⋆⋆ If you have not already done it, do Problem 3.11(b) and find the speed $v(t)$ of a rocket accelerating vertically from rest in a gravitational field g. Now integrate $v(t)$ and show that the rocket's height as a function of t is

$$y(t) = v_{\rm ex}t - \frac{1}{2}gt^2 - \frac{mv_{\rm ex}}{k}\ln\left(\frac{m_{\rm o}}{m}\right).$$

Using the numbers given in Problem 3.7, estimate the space shuttle's height after two minutes.

We can find the height $y(t)$ by integrating $v(t)$ as found in Problem 3.11:

$$\begin{aligned} y(t) &= \int_0^t v(t')dt' = v_{\rm ex}\int_0^t [\ln m_{\rm o} - \ln m(t')]dt' - \int_0^t gt'dt' \\ &= v_{\rm ex}t\ln m_{\rm o} - v_{\rm ex}\int_0^t \ln m(t')dt' - \tfrac{1}{2}gt^2. \end{aligned} \tag{S3.3}$$

To do the remaining integral, notice that $m(t') = m_{\rm o} - kt'$, so that $dt' = -dm'/k$, where m' is short for $m(t')$. Thus the remaining integral in (S3.3) is

$$\int_0^t \ln m'dt' = -\frac{1}{k}\int_{m_{\rm o}}^{m} \ln m'\,dm' = -\frac{1}{k}[m'\ln m' - m']_{m_{\rm o}}^{m} = \frac{1}{k}(m_{\rm o}\ln m_{\rm o} - m\ln m) - t,$$

where in the last expression I used the fact that $m_{\rm o} - m = kt$. Substituting into (S3.3), we get

$$y(t) = v_{\rm ex}t - \tfrac{1}{2}gt^2 + \frac{v_{\rm ex}}{k}(kt\ln m_{\rm o} - m_{\rm o}\ln m_{\rm o} + m\ln m) = v_{\rm ex}t - \tfrac{1}{2}gt^2 - \frac{mv_{\rm ex}}{k}\ln\left(\frac{m_{\rm o}}{m}\right)$$

where I have again used that $kt = m_{\rm o} - m$. Putting in the numbers from Problem 3.11 and $t = 2$ min, we find that $y \approx 40$ km.

SECTION 3.3 The Center of Mass

3.15 ⋆ Find the position of the center of mass of three particles lying in the xy plane at $\mathbf{r}_1 = (1, 1, 0)$, $\mathbf{r}_2 = (1, -1, 0)$, and $\mathbf{r}_3 = (0, 0, 0)$, if $m_1 = m_2$ and $m_3 = 10m_1$. Illustrate your answer with a sketch and comment.

$$X = \frac{\sum m_\alpha x_\alpha}{M} = \frac{1+1+0}{12} = \frac{1}{6}$$

$$Y = \frac{\sum m_\alpha y_\alpha}{M} = \frac{1-1+0}{12} = 0$$

$$Z = \frac{\sum m_\alpha z_\alpha}{M} = \frac{0+0+0}{12} = 0$$

Because m_3 is much bigger than m_1 and m_2, the CM is much closer to m_3 than to the other two.

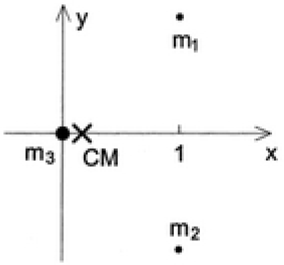


Figure S3.15

3.17 ⋆ The masses of the earth and moon are $M_e \approx 6.0 \times 10^{24}$ and $M_m \approx 7.4 \times 10^{22}$ (both in kg) and their center to center distance is 3.8×10^5 km. Find the position of their CM and comment. (The radius of the earth is $R_e \approx 6.4 \times 10^3$ km.)

If we put the origin at the earth's center and the x axis through the moon's center, then the CM lies on the x axis at

$$X = \frac{m_e x_e + m_m x_m}{m_e + m_m} = \frac{0 + m_m x_m}{m_e + m_m} = \frac{7.4 \times 10^{22}}{(600 + 7.4) \times 10^{22}} \times (3.8 \times 10^5 \text{ km}) = 4600 \text{ km}.$$

Since the earth's radius is 6400 km, the earth-moon CM is comfortably inside the earth, reflecting that the earth is much more massive than the moon.

3.19 ⋆⋆ **(a)** We know that the path of a projectile thrown from the ground is a parabola (if we ignore air resistance). In the light of the result (3.12), what would be the subsequent path of the CM of the pieces if the projectile exploded in midair? **(b)** A shell is fired from level ground so as to hit a target 100 m away. Unluckily the shell explodes prematurely and breaks into two equal pieces. The two pieces land at the same time, and one lands 100 m beyond the target. Where does the other piece land? **(c)** Is the same result true if they land at different times (with one piece still landing 100 m beyond the target)?

(a) As long as all pieces are still in the air (so that the only force on them is gravity), (3.11) implies that the CM follows the path of a single particle of mass M, namely, the same parabola.
(b) At the moment when both pieces land ($y = 0$), the CM is also at $y = 0$ and is, therefore, at the target ($x = 100$). Since the CM is half way between the two pieces and the first piece is 100 m beyond the target, the second piece has to be 100 m short of the target, namely back at the launch site ($x = 0$).
(c) Let's call the two halves A and B, and let's imagine the ground removed, so that the pieces can continue to move like projectiles even after they pass the level $y = 0$. At all times, the CM is at the midpoint of the line AB. Suppose that A lands first, at $x_A = 200$. At the same time, B and hence the CM are still above ground, which means that the CM has $X < 100$ (since we know the CM *will* land at $X = 100$). This implies that at the moment when A lands, $x_B < 0$. In other words, B has already passed back over the launch point and will land still further behind it. A similar argument shows that if A lands after B, then B will land at a point $x_B > 0$.

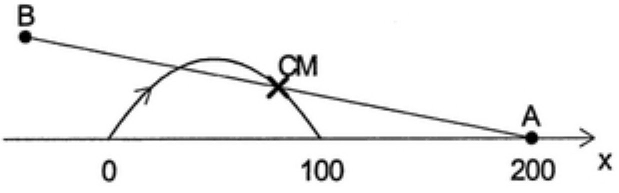


Figure S3.19

3.21 ⋆⋆ A uniform thin sheet of metal is cut in the shape of a semicircle of radius R and lies in the xy plane with its center at the origin and diameter lying along the x axis. Find the position of the CM using polar coordinates. [In this case the sum (3.9) that defines the CM position becomes a two-dimensional integral of the form $\int \mathbf{r}\,\sigma\,dA$ where σ denotes the surface mass density (mass/area) of the sheet and dA is the element of area $dA = r\,dr\,d\phi$.]

Let the disk's mass be M and its mass density (mass/area) be $\sigma = M/A$, where $A = \pi R^2/2$ is its area. The CM position is $\mathbf{R} = \int \sigma \mathbf{r}\,dA/M = \int \mathbf{r}\,dA/A$ where the integral runs over the area of the disk in the plane $z = 0$. Clearly $Z = 0$, and, by symmetry, $X = 0$. Finally

$$Y = \frac{1}{A}\int y\,dA = \frac{2}{\pi R^2}\int_0^R r\,dr \int_0^\pi d\phi(r\sin\phi) = \frac{4}{3\pi}R.$$

3.23 ⋆⋆⋆ [Computer] A grenade is thrown with initial velocity $\mathbf{v}_o$ from the origin at the top of a high cliff, subject to negligible air resistance. **(a)** Using a suitable plotting program, plot the orbit, with the following parameters: $\mathbf{v}_o = (4, 4)$, $g = 1$, and $0 \le t \le 4$ (and with x measured horizontally and y vertically up). Add to your plot suitable marks (dots or crosses, for example) to show the positions of the grenade at $t = 1, 2, 3, 4$. **(b)** At $t = 4$, when the grenade's velocity is $\mathbf{v}$, it explodes into two equal pieces, one of which moves off with velocity $\mathbf{v} + \Delta\mathbf{v}$. What is the velocity of the other piece? **(c)** Assuming that $\Delta\mathbf{v} = (1, 3)$, add to your original plot the paths of the two pieces for $4 \le t \le 9$. Insert marks to show their positions at $t = 5, 6, 7, 8, 9$. Find some way to show clearly that the CM of the two pieces continues to follow the original parabolic path.

(a) The orbit is $\mathbf{r}(t) = \mathbf{v}_o t + \frac{1}{2}\mathbf{g}t^2$.
(b) Because the two pieces have equal masses, $m_1 = m_2 = M/2$, conservation of momentum implies that $\mathbf{v}_1 + \mathbf{v}_2 = 2\mathbf{v}$, so that $\mathbf{v}_2 = 2\mathbf{v} - \mathbf{v}_1 = \mathbf{v} - \Delta\mathbf{v}$.
(c) The CM (hollow circles) is at the midpoint of the line joining the two pieces and clearly continues along the original parabola.

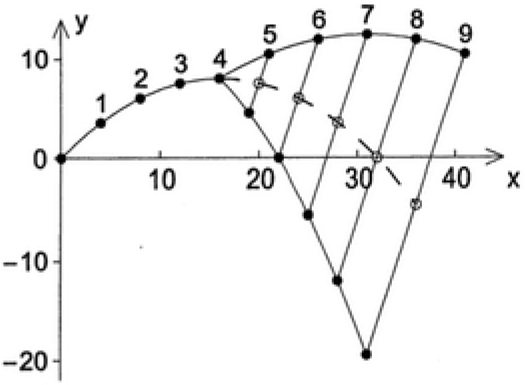


Figure S3.23

SECTION 3.4 Angular Momentum for a Single Particle

3.25 ⋆ A particle of mass m is moving on a frictionless horizontal table and is attached to a massless string, whose other end passes through a hole in the table, where I am holding it. Initially the particle is moving in a circle of radius r_o with angular velocity ω_o, but I now pull the string down through the hole until a length r remains between the hole and the particle. What is the particle's angular velocity now?

The net force on the particle is just the tension of the string, which is necessarily directed toward the hole in the table at O. Therefore the angular momentum $\boldsymbol{\ell}$ about O is constant. When the particle is travelling in a circle of radius r, the vertical component of $\boldsymbol{\ell} = \mathbf{r} \times \mathbf{p}$ is $\ell_z = rp = rmv = rm(r\omega) = mr^2\omega$. Therefore, the quantity $r^2\omega$ is constant and $r^2\omega = r_o^{\,2}\omega_o$; whence $\omega = (r_o/r)^2\omega_o$.

3.27 ⋆⋆ Consider a planet orbiting the fixed sun. Take the plane of the planet's orbit to be the xy plane, with the sun at the origin, and label the planet's position by polar coordinates (r, ϕ). **(a)** Show that the planet's angular momentum has magnitude $\ell = mr^2\omega$, where $\omega = \dot{\phi}$ is the planet's angular velocity about the sun. **(b)** Show that the rate at which the planet "sweeps out area" (as in Kepler's second law) is $dA/dt = \frac{1}{2}r^2\omega$, and hence that $dA/dt = \ell/2m$. Deduce Kepler's second law.

(a) Since $\mathbf{r} = r\hat{\mathbf{r}}$ and $\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}$, it follows that

$$\boldsymbol{\ell} = \mathbf{r} \times m\dot{\mathbf{r}} = mr^2\dot{\phi}\,\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} = mr^2\omega\hat{\mathbf{z}}$$

Therefore, $\ell = mr^2\omega$ as claimed. (Strictly speaking we should put absolute value signs in here if we are to insist that ℓ be positive.)
(b) Suppose that in a time Δt the planet moves from A to B and swings through an angle $\Delta\phi$. The area swept out is the area of the triangle OAB. In the limit of small Δt, this triangle is well approximated by the triangle OAC, with $OC = OA$. This has height r and base $r\Delta\phi$. Therefore the area swept out is $\Delta A \approx \frac{1}{2}r^2\Delta\phi$. Dividing both sides by Δt and letting $\Delta t \to 0$, we conclude that $\dot{A} = \frac{1}{2}r^2\dot{\phi} = \frac{1}{2}r^2\omega$, as claimed. Comparing this with the result of part (a), we see that $\dot{A} = \ell/2m$ and hence that the conservation of ℓ implies that $\dot{A}$ is constant.

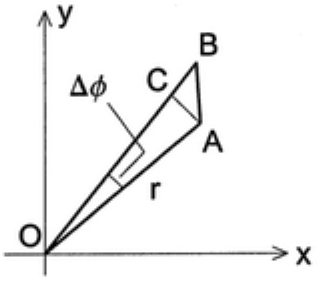


Figure S3.27

SECTION 3.5 Angular Momentum for Several Particles

3.29 ⋆ A uniform spherical asteroid of radius R_o is spinning with angular velocity ω_o. As the aeons go by, it picks up more matter until its radius is R. Assuming that its density remains the same and that the additional matter was originally at rest relative to the asteroid (anyway on average), find the asteroid's new angular velocity. (You know from elementary physics that the moment of inertia is $\frac{2}{5}MR^2$.) What is the final angular velocity if the radius doubles?

We are told that the matter accreted by the asteroid is initially at rest. Therefore its initial angular momentum is zero and, by conservation of angular momentum, $I_o\omega_o = I\omega$, where I_o and I are the initial and final moments of inertia of the asteroid. Now,

$$I = \frac{2}{5}MR^2 = \frac{2}{5}\left(\frac{4}{3}\pi\varrho R^3\right)R^2 = \frac{8}{15}\pi\varrho R^5,$$

so, given that ϱ is constant, conservation of angular momentum implies that $R_o^5\omega_o = R^5\omega$, and $\omega = \omega_o(R_o/R)^5$. If $R = 2R_o$, then $\omega = \omega_o/32$.

3.30 ⋆⋆ Consider a rigid body rotating with angular velocity ω about a fixed axis. (You could think of a door rotating about the axis defined by its hinges.) Take the axis of rotation to be the z axis and use cylindrical polar coordinates $\rho_\alpha, \phi_\alpha, z_\alpha$ to specify the positions of the particles $\alpha = 1, \cdots, N$ that make up the body. **(a)** Show that the velocity of the particle α is $\rho_\alpha\omega$ in the ϕ direction. **(b)** Hence show that the z component of the angular momentum ℓ_α of particle α is $m_\alpha\rho_\alpha^2\omega$. **(c)** Show that the

z component L_z of the total angular momentum can be written as $L_z = I\omega$ where I is the moment of inertia (for the axis in question),

$$I = \sum_{\alpha=1}^{N} m_\alpha \rho_\alpha^2. \tag{3.31}$$

3.31 ⋆⋆ Find the moment of inertia of a uniform disc of mass M and radius R rotating about its axis, by replacing the sum (3.31) by the appropriate integral and doing the integral in polar coordinates.

If we place the disk in the plane $z = 0$ centered on the origin, the sum (3.31) can be written as $I = \sum m_\alpha r_\alpha^2$ (because in the plane $z = 0$ the coordinate ρ is the same as what we usually call r). If the density (mass/area) of the disk is $\sigma = M/\pi R^2$, then, replacing the sum by the appropriate integral, we find that

$$I = \int \sigma r^2 dA = \frac{M}{\pi R^2} \int_0^R r\,dr \int_0^{2\pi} d\phi\, r^2 = \frac{M}{\pi R^2} \cdot \frac{R^4}{4} \cdot 2\pi = \tfrac{1}{2} M R^2$$

3.33 ⋆⋆ Starting from the sum (3.31) and replacing it by the appropriate integral, find the moment of inertia of a uniform thin square of side $2b$, rotating about an axis perpendicular to the square and passing through its center.

If we place the square in the plane $z = 0$, centered on the origin with its sides parallel to the x and y axes, the sum (3.31) takes the form $I = \sum m_\alpha \rho_\alpha^2 = \sum m_\alpha (x_\alpha^2 + y_\alpha^2) = 2 \sum m_\alpha x_\alpha^2$, where the last expression holds because, by symmetry, the two terms of the previous expression are equal. If we denote the density (mass/area) of the square by $\sigma = M/A = M/(4b^2)$, then we can replace the sum by an integral:

$$I = 2\sigma \int x^2 dA = 2\sigma \int_{-b}^{b} x^2 dx \int_{-b}^{b} dy = 2\frac{M}{4b^2} \cdot \frac{2b^3}{3} \cdot 2b = \frac{2}{3} M b^2$$

3.35 ⋆⋆ Consider a uniform solid disk of mass M and radius R, rolling without slipping down an incline which is at angle γ to the horizontal. The instantaneous point of contact between the disk and the incline is called P. **(a)** Draw a free-body diagram, showing all forces on the disk. **(b)** Find the linear acceleration $\dot{v}$ of the disk by applying the result $\dot{\mathbf{L}} = \mathbf{\Gamma}^{\text{ext}}$ for rotation about P. (Remember that $L = I\omega$ and the moment of inertia for rotation about a point on the circumference is $\frac{3}{2} M R^2$. The condition that the disk not slip is that $v = R\omega$ and hence $\dot{v} = R\dot{\omega}$.) **(c)** Derive the same result by applying $\dot{\mathbf{L}} = \mathbf{\Gamma}^{\text{ext}}$ to the rotation about the CM. (In this case you will find there is an extra unknown, the force of friction. You can eliminate this by applying Newton's second law to the motion of the CM. The moment of inertia for rotation about the CM is $\frac{1}{2} M R^2$.)

(a)

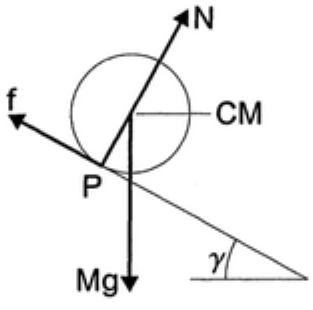


Figure S3.35

(b) The condition $\dot{\mathbf{L}} = \mathbf{\Gamma}^{\text{ext}}$ applied about P becomes $I_P\,\dot{\omega} = MgR\sin\gamma$, whence $\dot{v} = \frac{2}{3}g\sin\gamma$.
(c) The same condition applied about the CM gives $I_{\text{cm}}\dot{\omega} = fR$. To eliminate the unknown frictional force f, we must use Newton's second law, $M\dot{v} = Mg\sin\gamma - f$. Eliminating f, we get the same answer as before.

3.37 ★★★ A system consists of N masses m_α at positions $\mathbf{r}_\alpha$ relative to a fixed origin O. Let $\mathbf{r}'_\alpha$ denote the position of m_α relative to the CM; that is, $\mathbf{r}'_\alpha = \mathbf{r}_\alpha - \mathbf{R}$. **(a)** Make a sketch to illustrate this last equation. **(b)** Prove the useful relation that $\sum m_\alpha\mathbf{r}'_\alpha = 0$. Can you explain why this relation is nearly obvious? **(c)** Use this relation to prove the result (3.28) that the rate of change of the angular momentum *about the CM* is equal to the total external torque about the CM. (This result is surprising since the CM may be accelerating, so that it is not necessarily a fixed point in any inertial frame.)

(a)

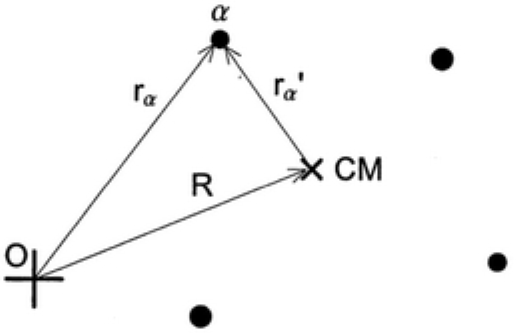


Figure S3.37

(b) $\sum m_\alpha\mathbf{r}'_\alpha = \sum m_\alpha(\mathbf{r}_\alpha - \mathbf{R}) = \sum m_\alpha\mathbf{r}_\alpha - (\sum m_\alpha)\mathbf{R} = M\mathbf{R} - M\mathbf{R} = 0.$
The sum $(1/M)\sum m_\alpha\mathbf{r}'_\alpha$ defines the position of the CM *relative to the CM*, which is fairly obviously zero.
(c) The angular momentum about the CM is $\mathbf{L}(\text{about CM}) = \sum \mathbf{r}'_\alpha \times m_\alpha\dot{\mathbf{r}}'_\alpha$. Therefore

$$\begin{aligned}\dot{\mathbf{L}}(\text{about CM}) &= \sum \dot{\mathbf{r}}'_\alpha \times m_\alpha\dot{\mathbf{r}}'_\alpha + \sum \mathbf{r}'_\alpha \times m_\alpha\ddot{\mathbf{r}}'_\alpha \\ &= 0 + \sum \mathbf{r}'_\alpha \times m_\alpha(\ddot{\mathbf{r}}_\alpha - \ddot{\mathbf{R}}) \\ &= \sum \mathbf{r}'_\alpha \times F_\alpha - \sum m_\alpha\mathbf{r}'_\alpha \times \ddot{\mathbf{R}} \\ &= \mathbf{\Gamma}(\text{about CM}) + 0 = \mathbf{\Gamma}^{\text{ext}}(\text{about CM}).\end{aligned}$$

The first sum on the right of the top line was zero because the cross product of any two parallel vectors is zero. The second sum on the third line was zero by the result of part (b), and, in the last line, all internal torques canceled for the same reason as in (3.25).

CHAPTER 4

Energy

PROBLEMS AND SOLUTIONS

SECTION 4.1 Kinetic Energy and Work

4.1 ⋆ By writing **a** · **b** in terms of components prove that the product rule for differentiation applies to the dot product of two vectors; that is,

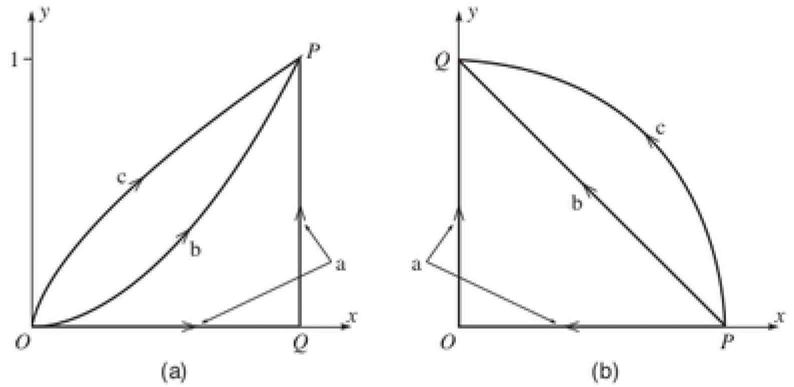
$$\frac{d}{dt}(\mathbf{a}\cdot\mathbf{b}) = \frac{d\mathbf{a}}{dt}\cdot\mathbf{b} + \mathbf{a}\cdot\frac{d\mathbf{b}}{dt}.$$

$$\frac{d}{dt}(\mathbf{a}\cdot\mathbf{b}) = \frac{d}{dt}\sum a_i b_i = \sum\left(a_i\frac{db_i}{dt} + \frac{da_i}{dt}b_i\right) = \mathbf{a}\cdot\frac{d\mathbf{b}}{dt} + \frac{d\mathbf{a}}{dt}\cdot\mathbf{b}.$$

4.2 ⋆⋆ Evaluate the work done

$$W = \int_O^P \mathbf{F}\cdot d\mathbf{r} = \int_O^P (F_x\,dx + F_y\,dy) \tag{4.100}$$

by the two-dimensional force $\mathbf{F} = (x^2, 2xy)$ along the three paths joining the origin to the point $P = (1, 1)$ as shown in Figure 4.24(a) and defined as follows: **(a)** This path goes along the x axis to $Q = (1, 0)$ and then straight up to P. (Divide the integral into two pieces, $\int_O^P = \int_O^Q + \int_Q^P$.) **(b)** On this path $y = x^2$, and you can replace the term dy in (4.100) by $dy = 2x\,dx$ and convert the whole integral into an integral over x. **(c)** This path is given parametrically as $x = t^3$, $y = t^2$. In this case rewrite x, y, dx, and dy in (4.100) in terms of t and dt, and convert the integral into an integral over t.



(a) (b)

Figure 4.24 **(a)** Problem 4.2. **(b)** Problem 4.3

4.3 ★★ Do the same as in Problem 4.2, but for the force $\mathbf{F} = (-y, x)$ and for the three paths joining P and Q shown in Figure 4.24(b) and defined as follows: **(a)** This path goes straight from $P = (1, 0)$ to the origin and then straight to $Q = (0, 1)$. **(b)** This is a straight line from P to Q. (Write y as a function of x and rewrite the integral as an integral over x.) **(c)** This is a quarter-circle centered on the origin. (Write x and y in polar coordinates and rewrite the integral as an integral over ϕ.)

(a)

$$W = \int_P^Q \mathbf{F} \cdot d\mathbf{r} = \int_P^O F_x dx + \int_O^Q F_y dy = 0 + 0 = 0.$$

(b) On path (b), $y = 1 - x$, so $dy = -dx$ and

$$W = \int_P^Q (F_x dx + F_y dy) = \int_1^0 (F_x - F_y) dx = \int_0^1 (y + x) dx = \int_0^1 1 dx = 1.$$

(c) On path (c), $\mathbf{r} = (\cos\phi, \sin\phi)$, so $\mathbf{F} = (-\sin\phi, \cos\phi)$ and $d\mathbf{r} = (-\sin\phi, \cos\phi)d\phi$. Therefore

$$W = \int_P^Q \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} (\sin^2\phi + \cos^2\phi) d\phi = \pi/2.$$

SECTION 4.2 Potential Energy and Conservative Forces

4.5 ★ **(a)** Consider a mass m in a uniform gravitational field $\mathbf{g}$, so that the force on m is $m\mathbf{g}$, where $\mathbf{g}$ is a constant vector pointing vertically down. If the mass moves by an arbitrary path from point 1 to point 2, show that the work done by gravity is $W_{\text{grav}}(1 \to 2) = -mgh$ where h is the vertical height gained between points 1 and 2. Use this result to prove that the force of gravity is conservative (at least in a region small enough so that $\mathbf{g}$ can be considered constant). **(b)** Show that, if we choose axes with y measured vertically up, the gravitational potential energy is $U = mgy$ (if we choose $U = 0$ at the origin).

(a) If we choose axes with y measured vertically up (and x and z horizontal), then $\mathbf{F}_{\text{grav}} = (0, mg, 0)$ and

$$W_{\text{grav}}(1 \to 2) = \int_1^2 \mathbf{F}_{\text{grav}} \cdot d\mathbf{r} = -\int_1^2 mg\, dy = -mg(y_2 - y_1) = -mgh.$$

Since this is visibly independent of the path followed, $\mathbf{F}_{\text{grav}}$ is conservative.
(b) $U_{\text{grav}}(\mathbf{r}) = -W_{\text{grav}}(0 \to \mathbf{r}) = mgy.$

4.7 ★ Near to the point where I am standing on the surface of Planet X, the gravitational force on a mass m is vertically down but has magnitude $m\gamma y^2$ where γ is a constant and y is the mass's height above the horizontal ground. **(a)** Find the work done by gravity on a mass m moving from $\mathbf{r}_1$ to $\mathbf{r}_2$, and use your answer to show that gravity on Planet X, although most unusual, is still conservative. Find the corresponding potential energy. **(b)** Still on the same planet, I thread a bead on a curved, frictionless, rigid wire, which extends from ground level to a height h above the ground. Show clearly in a picture the forces on the bead when it is somewhere on the wire. (Just name the forces so it's clear what they are; don't worry about their magnitude.) Which of the forces are conservative and which are not? **(c)** If I release the bead from rest at a height h, how fast will it be going when it reaches the ground?

(a) Since $\mathbf{F}_{\text{gr}} = -m\gamma y^2 \hat{\mathbf{y}}$,

$$W(\mathbf{r}_1 \to \mathbf{r}_2) = \int_1^2 \mathbf{F}_{\text{gr}} \cdot d\mathbf{r} = -m\gamma \int_{y_1}^{y_2} y^2 dy = -\frac{m\gamma}{3}({y_2}^3 - {y_1}^3).$$

Because this work is independent of path (and $\mathbf{F}_{\text{gr}}$ depends on only $\mathbf{r}$) $\mathbf{F}_{\text{gr}}$ is conservative, and

$$U(\mathbf{r}) = -W(0 \to \mathbf{r}) = \frac{m\gamma}{3} y^3.$$

(b) The forces are $\mathbf{F}_{\text{gr}}$, which is conservative, and the normal force $\mathbf{N}$, which is not. (For instance, $\mathbf{N}$ can depend on the bead's speed.)

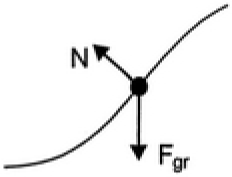


Figure S4.7

(c) By conservation of energy, $T + U = \text{const}$, so $\frac{1}{2}m{v_{\text{fin}}}^2 = \frac{1}{3}m\gamma h^3$ and hence

$$v_{\text{fin}} = \sqrt{2\gamma h^3/3}.$$

4.9 ⋆⋆ **(a)** The force exerted by a one-dimensional spring, fixed at one end, is $F = -kx$, where x is the displacement of the other end from its equilibrium position. Assuming that this force is conservative (which it is) show that the corresponding potential energy is $U = \frac{1}{2}kx^2$, if we choose U to be zero at the equilibrium position. **(b)** Suppose that this spring is hung vertically from the ceiling with a mass m suspended from the other end and constrained to move in the vertical direction only. Find the extension x_o of the new equilibrium position with the suspended mass. Show that the total potential energy (spring plus gravity) has the same form $\frac{1}{2}ky^2$ if we use the coordinate y equal to the displacement measured from the new equilibrium position at $x = x_o$ (and redefine our reference point so that $U = 0$ at $y = 0$).

(a)

$$U = -\int_0^x F(x')dx' = k\int_0^x x'dx' = \tfrac{1}{2}kx^2$$

(b) The two forces on m are gravity, mg down, and the spring, kx up, and at the new equilibrium x_o they must balance, so $x_o = mg/k$. The total PE is

$$\begin{aligned} U &= \tfrac{1}{2}kx^2 - mgx \\ &= \tfrac{1}{2}k(x_o + y)^2 - mg(x_o + y) \\ &= \tfrac{1}{2}ky^2 + y(kx_o - mg) + \text{ const.} \end{aligned}$$

In the last line the second term is zero (the condition for equilibrium) and we can drop the constant, to give $U = \frac{1}{2}ky^2$.

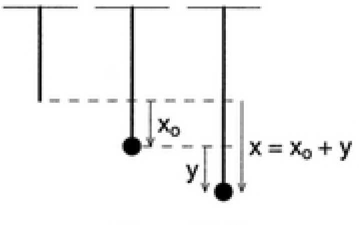


Figure S4.9

SECTION 4.3 Force as the Gradient of Potential Energy

4.11 ⋆ Find the partial derivatives with respect to x, y, and z of the following functions: **(a)** $f(x, y, z) = ay^2 + 2byz + cz^2$, **(b)** $g(x, y, z) = \cos(axy^2z^3)$, **(c)** $h(x, y, z) = ar$, where a, b, and c are constants and $r = \sqrt{x^2 + y^2 + z^2}$. Remember that to evaluate $\partial f/\partial x$ you differentiate with respect to x treating y and z as constants.

function, f	$\partial f/\partial x$	$\partial f/\partial y$	$\partial f/\partial z$
$ay^2 + 2byz + cz^2$	0	$2ay + 2bz$	$2by + 2cz$
$\cos(axy^2z^3)$	$-ay^2z^3\sin(axy^2z^3)$	$-2axyz^3\sin(axy^2z^3)$	$-3axy^2z^2\sin(axy^2z^3)$
$ar = a\sqrt{x^2 + y^2 + z^2}$	ax/r	ay/r	az/r

4.13 ⋆ Calculate the gradient ∇f of the following functions, $f(x, y, z)$: **(a)** $f = \ln(r)$, **(b)** $f = r^n$, **(c)** $f = g(r)$, where $r = \sqrt{x^2 + y^2 + z^2}$ and $g(r)$ is some unspecified function of r. [*Hint:* Use the chain rule.]

function, f	$\partial f/\partial x$	$\partial f/\partial y$	$\partial f/\partial z$	∇f
$\ln(r)$	x/r^2	y/r^2	z/r^2	$\hat{\mathbf{r}}/r$
r^n	nxr^{n-2}	nyr^{n-2}	nzr^{n-2}	$n\,r^{n-1}\hat{\mathbf{r}}$
$g(r)$	$g'(r)x/r$	$g'(r)y/r$	$g'(r)z/r$	$g'(r)\,\hat{\mathbf{r}}$

4.15 ⋆ For $f(\mathbf{r}) = x^2 + 2y^2 + 3z^2$, use the approximation (4.35) to estimate the change in f if we move from the point $\mathbf{r} = (1, 1, 1)$ to $(1.01, 1.03, 1.05)$. Compare with the exact result.

If $f(\mathbf{r}) = x^2 + 2y^2 + 3z^2$, the gradient is $\nabla f = (2x, 4y, 6z)$, so Equation (4.35) gives the approximation

$$\begin{aligned} df \approx \nabla f \cdot d\mathbf{r} &= (2x, 4y, 6z) \cdot (dx, dy, dz) \\ &= (2, 4, 6) \cdot (0.01, 0.03, 0.05) = 0.02 + 0.12 + 0.30 = 0.4400. \end{aligned}$$

This compares favorably with the exact result

$$df = f(1.01, 1.03, 1.05) - f(1, 1, 1) = 6.4494 - 6.0000 = 0.4494.$$

4.17 ⋆ A charge q in a uniform electric field $\mathbf{E}_o$ experiences a constant force $\mathbf{F} = q\mathbf{E}_o$. **(a)** Show that this force is conservative and verify that the potential energy of the charge at position $\mathbf{r}$ is $U(\mathbf{r}) = -q\mathbf{E}_o \cdot \mathbf{r}$. **(b)** By doing the necessary derivatives, check that $\mathbf{F} = -\nabla U$.

(a) The force $\mathbf{F} = q\mathbf{E}_o$ certainly doesn't depend on anything but $\mathbf{r}$ (and doesn't even depend on $\mathbf{r}$ since $\mathbf{E}_o$ is constant). The work integral is $W(1 \to 2) = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = q\mathbf{E}_o \cdot \int_1^2 d\mathbf{r} = q\mathbf{E}_o \cdot (\mathbf{r}_2 - \mathbf{r}_1)$, which is independent of path. Therefore, the force is conservative, and the PE is $U(\mathbf{r}) = -W(0 \to \mathbf{r}) = -q\mathbf{E}_o \cdot \mathbf{r}$.
(b) Since $U = -q(E_{ox}x + E_{oy}y + E_{oz}z)$, $-\nabla U = q(E_{ox}, E_{oy}, E_{oz}) = q\mathbf{E}_o = \mathbf{F}$.

4.19 ⋆⋆ **(a)** Describe the surfaces defined by the equation $f =$ const, where $f = x^2 + 4y^2$. **(b)** Using the results of Problem 4.18, find a unit normal to the surface $f = 5$ at the point $(1, 1, 1)$. In what direction should one move from this point to maximize the rate of change of f?

(a) The equation $x^2 + 4y^2 = K$ defines an ellipse in the xy plane, centered on $x = y = 0$. Since the value of $x^2 + 4y^2$ is unchanged by any variation of z, the surface is an elliptical cylinder, centered on the z axis, with semi-major and semi-minor axes equal to $\sqrt{K}$ and $\sqrt{K}/2$.
(b) We know that the vector ∇f is perpendicular to the surface f = const. In the present case $\nabla f = (2x, 8y, 0) = (2, 8, 0)$ at the point $(1, 1, 1)$. Therefore the unit normal is $\mathbf{n} = (1, 4, 0)/\sqrt{17}$ or $-\mathbf{n}$. The direction of maximum increase is $\mathbf{n}$.

SECTION 4.4 The Second Condition that F be Conservative

4.21 ⋆ Verify that the gravitational force $-GMm\hat{\mathbf{r}}/r^2$ on a point mass m at $\mathbf{r}$, due to a fixed point mass M at the origin, is conservative and calculate the corresponding potential energy.

This is exactly the same as Example 4.5 except that $\gamma = kqQ$ is replaced by $\gamma = -GMm$. Thus, if we choose $U = 0$ at $r = \infty$, then $U(r) = \gamma/r = -GMm/r$.

4.23 ⋆⋆ Which of the following forces is conservative? **(a)** $\mathbf{F} = k(x, 2y, 3z)$ where k is a constant. **(b)** $\mathbf{F} = k(y, x, 0)$. **(c)** $\mathbf{F} = k(-y, x, 0)$. For those which are conservative, find the corresponding potential energy U, and verify by direct differentiation that $\mathbf{F} = -\nabla U$.

All three given forces depend only on $\mathbf{r}$; that is, all satisfy the first condition. It remains to checks their curls.
(a) With $\mathbf{F} = k(x, 2y, 3z)$, we find $\nabla \times \mathbf{F} = (0, 0, 0)$, so $\mathbf{F}$ is conservative. The corresponding PE is $U = -\int \mathbf{F} \cdot d\mathbf{r}' = -k \int (x'dx' + 2y'dy' + 3z'dz') = -k(\frac{1}{2}x^2 + y^2 + \frac{3}{2}z^2)$. Clearly $-\nabla U = k(x, 2y, 3z) = \mathbf{F}$. (I have taken $U = 0$ at the origin.)
(b) With $\mathbf{F} = k(y, x, 0)$, we find $\nabla \times \mathbf{F} = k(0, 0, 1 - 1) = (0, 0, 0)$, so $\mathbf{F}$ is conservative. The corresponding PE is $U = -\int_0^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r}' = -k \int_0^{\mathbf{r}} (y'dx' + x'dy')$. This integral can be evaluated along any path. One simple choice is to go from the origin out to x on the x axis (this contributes 0 to the integral) and then parallel to the y axis to y (which contributes $-kxy$). Thus $U = -kxy$. Clearly $-\nabla U = k(y, x, 0) = \mathbf{F}$.
(c) With $\mathbf{F} = k(-y, x, 0)$, we find $\nabla \times \mathbf{F} = k(0, 0, 1 + 1) = k(0, 0, 2)$, so $\mathbf{F}$ is not conservative.

4.25 ⋆⋆⋆ The proof that the condition $\nabla \times \mathbf{F} = 0$ guarantees the path independence of the work $\int_1^2 \mathbf{F} \cdot d\mathbf{r}$ done by $\mathbf{F}$ is unfortunately too lengthy to be included here. However, the following three exercises capture the main points:[1] **(a)** Show that the path independence of $\int_1^2 \mathbf{F} \cdot d\mathbf{r}$ is equivalent to the statement that the integral $\oint_\Gamma \mathbf{F} \cdot d\mathbf{r}$ around any *closed* path Γ is zero. (By tradition, the symbol $\oint$ is used for integrals around a closed path — a path that starts and stops at the same point.) [*Hint:* For any two points 1 and 2 and any two paths from 1 to 2, consider the work done by $\mathbf{F}$ going from 1 to 2 along the first path and then back to 1 along the second in the reverse direction.] **(b)** Stokes's theorem asserts that $\oint_\Gamma \mathbf{F} \cdot d\mathbf{r} = \int (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dA$, where the integral on the right is a surface integral over a surface for which the path Γ is the boundary, and $\hat{\mathbf{n}}$ and dA are a unit normal to the surface and an element of area. Show that Stokes's theorem implies that if $\nabla \times \mathbf{F} = 0$ everywhere, then $\oint_\Gamma \mathbf{F} \cdot d\mathbf{r} = 0$. **(c)** While the general proof of Stokes's theorem is beyond our scope here, the following special case is quite easy to prove (and is an important step toward the general proof): Let Γ denote a rectangular closed path lying in a plane perpendicular to the z direction and bounded by the lines $x = B$, $x = B + b$, $y = C$ and $y = C + c$. For this simple path (traced counterclockwise as seen from above), prove Stokes's theorem that

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[1] For a complete discussion see, for example, *Mathematical Methods,* Boas, Ch. 6, Sections 8–11.

$$\oint_{\Gamma} \mathbf{F} \cdot d\mathbf{r} = \int (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dA$$

where $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ and the integral on the right runs over the flat, rectangular area inside Γ. [*Hint:* The integral on the left contains four terms, two of which are integrals over x and two over y. If you pair them in this way, you can combine each pair into a single integral with an integrand of the form $F_x(x, C + c, z) - F_x(x, C, z)$ (or a similar term with the roles of x and y exchanged). You can rewrite this integrand as an integral over y of $\partial F_x(x, y, z)/\partial y$ (and similarly with the other term), and you're home.]

(a) Let 1 and 2 denote any two points and Γ_a and Γ_b be any two paths leading from point 1 to point 2. Next let Γ be the closed path that starts at point 1, goes to point 2 via Γ_a, and then returns to point 1 tracing the path Γ_b backwards. Obviously

$$\oint_{\Gamma} \mathbf{F} \cdot d\mathbf{r} = \int_{\Gamma_a} \mathbf{F} \cdot d\mathbf{r} - \int_{\Gamma_b} \mathbf{F} \cdot d\mathbf{r}$$

The work integral is path-independent if and only if the right side is zero for any two paths joining any two points 1 and 2, and the left side is zero if and only if the work integral is zero around any closed path. Therefore the two statements are equivalent.

(b) If we accept Stokes's theorem, $\oint_{\Gamma} \mathbf{F} \cdot d\mathbf{r} = \int (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dA$, then obviously $\oint_{\Gamma} \mathbf{F} \cdot d\mathbf{r} = 0$ if $\nabla \times \mathbf{F} = 0$ everywhere.

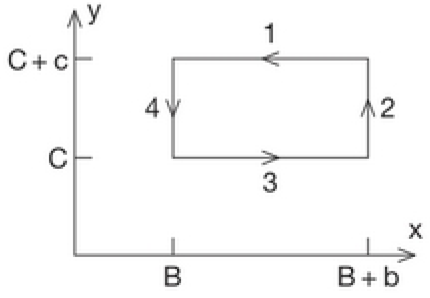

Figure S4.25

(c) The integral going around the closed path Γ can be divided into four integrals, each along one of the straight paths labelled 1, 2, 3, and 4 in the picture.

$$\oint_{\Gamma} \mathbf{F} \cdot d\mathbf{r} = \left(\int_1 + \int_2 + \int_3 + \int_4 \right) \mathbf{F} \cdot d\mathbf{r}$$

Now

$$\int_1 + \int_3 = -\int_B^{B+b} F_x(x, C + c, z) dx + \int_B^{B+b} F_x(x, C, z) dx$$

and

$$F_x(x, C + c, z) dx - F_x(x, C, z) = \int_C^{C+c} dy\, \frac{\partial F_x(x, y, z)}{\partial y}.$$

Combining the last two results, we find that

$$\int_1 + \int_3 = -\int_B^{B+b} dx \int_C^{C+c} dy\, \frac{\partial F_x(x, y, z)}{\partial y} = -\int \frac{\partial F_x}{\partial y} dA$$

where the final integral is a two-dimensional integral over the whole rectangle. There is a similar expression (without a minus sign) for $\int_2 + \int_4$, and, combining these two, we conclude that

$$\oint_\Gamma \mathbf{F} \cdot d\mathbf{r} = \int \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dA = \int (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dA.$$

SECTION 4.5 Time-Dependent Potential Energy

4.27 ⋆⋆ Suppose that the force $\mathbf{F}(\mathbf{r}, t)$ depends on the time t but still satisfies $\nabla \times \mathbf{F} = 0$. It is a mathematical fact (related to Stokes's theorem as discussed in Problem 4.25) that the work integral $\int_1^2 \mathbf{F}(\mathbf{r}, t) \cdot d\mathbf{r}$ (evaluated at any one time t) is independent of the path taken between the points 1 and 2. Use this to show that the time-dependent PE defined by (4.48), for any fixed time t, has the claimed property that $\mathbf{F}(\mathbf{r}, t) = -\nabla U(\mathbf{r}, t)$. Can you see what goes wrong with the argument leading to Equation (4.19), that is, conservation of energy?

Stokes's theorem guarantees that the integral (4.48) is path-independent and hence that it defines a unique function $U(\mathbf{r}, t)$. To show that $-\nabla U = \mathbf{F}$, consider the difference between the values of U at two neighboring points $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$:

$$dU = U(\mathbf{r} + d\mathbf{r}, t) - U(\mathbf{r}, t) \qquad \text{(S4.1)}$$

$$= -\int_{\mathbf{r}}^{\mathbf{r}+d\mathbf{r}} \mathbf{F}(\mathbf{r}', t) \cdot d\mathbf{r}' = -\mathbf{F}(\mathbf{r}, t) \cdot d\mathbf{r}. \qquad \text{(S4.2)}$$

But $dU = (\nabla U) \cdot d\mathbf{r}$, and since both of these expressions for dU are valid for all (small) $d\mathbf{r}$, it follows that $\mathbf{F}(\mathbf{r}, t) = -\nabla U(\mathbf{r}, t)$. The argument leading to Eq. (4.19) requires that we look at the change in $E = T + U$ as we follow the particle moving from $\mathbf{r}$ to $\mathbf{r} + d\mathbf{r}$ as the time advances from t to $t + dt$. The change in T is $dT = \mathbf{F} \cdot d\mathbf{r}$. The change in U is

$$dU = U(\mathbf{r} + d\mathbf{r}, t + dt) - U(\mathbf{r}, t) \qquad \text{(S4.3)}$$

and here's the difficulty. This dU is not the same as the dU in Eq. (S4.1). Here we are concerned with the change dU as the particle moves from $\mathbf{r}$ to $\mathbf{r} + d\mathbf{r}$ *and* the time changes from t to $t + dt$. In Eq. (S4.1) dU is the difference between the values at $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$ *at the same time* t. Thus the difference (S4.3) is not the same as $-\mathbf{F}(\mathbf{r}, t) \cdot d\mathbf{r}$ as in (S4.2); it does not cancel the change in KE when we evaluate $dE = dT + dU$, and mechanical energy is not conserved.

SECTION 4.6 Energy for Linear One-Dimensional Systems

4.29 ⋆⋆ [Computer] A mass m confined to the x axis has potential energy $U = kx^4$ with $k > 0$. **(a)** Sketch this potential energy and qualitatively describe the motion if the mass is initially stationary at $x = 0$ and is given a sharp kick to the right at $t = 0$. **(b)** Use (4.58) to find the time for the mass to reach its maximum displacement $x_{\max} = A$. Give your answer as an integral over x in terms of m, A, and k. Hence find the period τ of oscillations of amplitude A as an integral. **(c)** By making a suitable change of variables in the integral, show that the period τ is inversely proportional to the amplitude A. **(d)** The integral of part (b) cannot be evaluated in terms of elementary functions, but it can be done numerically. Find the period for the case that $m = k = A = 1$.

(a) The mass moves to the right to the turning point at $x = A$, where $E = U = kA^4$ so that T and hence v are zero. It speeds up as it moves back to O and then moves out to the left until $x = -A$, where $E = U$ again. It then moves back to O, and repeats the whole cycle indefinitely (in the absence of any friction). Notice how this quartic well is much flatter at the bottom than the parabolic harmonic well.

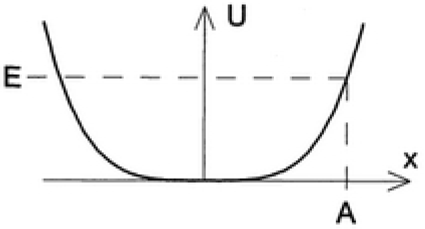

Figure S4.29

(b) Since $T = E - U = k(A^4 - x^4)$, the velocity is $\dot{x} = \sqrt{2k/m}\sqrt{A^4 - x^4}$ (moving to the right). According to (4.58), the time to move out to $x = A$ is

$$t(0\to A) = \int_0^A \frac{dx}{\dot{x}} = \sqrt{\frac{m}{2k}} \int_0^A \frac{dx}{\sqrt{A^4 - x^4}}.$$

(c) The period is four times this time. Thus, if we change variables to $u = x/A$,

$$\tau = \frac{1}{A}\sqrt{\frac{8m}{k}} \int_0^1 \frac{du}{\sqrt{1 - u^4}}$$

which is inversely proportional to A as claimed.
(d) Using any suitable software, the integral can be found to be 1.31, and setting $m = k = A = 1$ we find $\tau = 3.71$.

SECTION 4.7 Curvilinear One-Dimensional Systems

4.31 ⋆ **(a)** Write down the total energy E of the two masses in the Atwood machine of Figure 4.15 in terms of the coordinate x and $\dot{x}$. **(b)** Show (what is true for any conservative one-dimensional system) that you can obtain the equation of motion for the coordinate x by differentiating the equation $E =$ const. Check that the equation of motion is the same as you would obtain by applying Newton's second law to each mass and eliminating the unknown tension from the two resulting equations.

(a) Because the string is inextensible, the height of m_2 below the wheel is $y = k - x$, where k is a constant, and its velocity is $\dot{y} = -\dot{x}$. Thus the total energy is $E = T + U = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 - m_1gx - m_2g(k - x)$ or $E = \frac{1}{2}(m_1 + m_2)\dot{x}^2 - (m_1 - m_2)gx$, if we drop the uninteresting constant $-m_2gk$.
(b) The equation $dE/dt = 0$ yields $(m_1 + m_2)\dot{x}\ddot{x} - (m_1 - m_2)g\dot{x} = 0$, or

$$(m_1 + m_2)\ddot{x} = (m_1 - m_2)g. \tag{S4.4}$$

Applying the second law to each separate mass, we find

$$m_1\ddot{x} = m_1g - F_T \qquad \text{and} \qquad m_2\ddot{x} = F_T - m_2g$$

where F_T is the tension in the string. Adding these two equations, we can eliminate the tension and we get precisely Eq. (S4.4).

4.33 ⋆⋆ [Computer] **(a)** Verify the expression (4.59) for the potential energy of the cube balanced on a cylinder in Example 4.7 (page 130). **(b)** Make plots of $U(\theta)$ for $b = 0.9r$ and $b = 1.1r$. (You may as well choose units such that r, m, and g are all equal to 1.) **(c)** Use your plots to confirm the findings of Example 4.7 concerning the stability of the equilibrium at $\theta = 0$. Are there any other equilibrium points and are they stable?

(a) The derivation of $U(\theta)$ is given above Eq. (4.59).
(b)

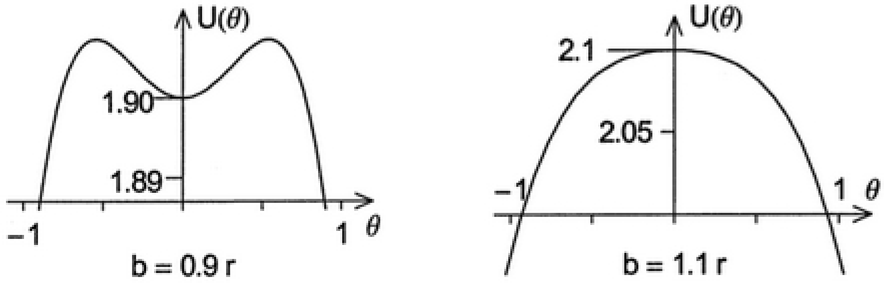

Figure S4.33

(c) These two plots do bear out the finding of Example 4.7 that for $b < r$ the equilibrium at $\theta = 0$ is stable, whereas for $b > r$ it is unstable. In addition, they show that for $b < r$ there can be two further equilibrium points (symmetrically placed on either side of $\theta = 0$), both of which are unstable.

4.35 ⋆⋆ Consider the Atwood machine of Figure 4.15, but suppose that the pulley has radius R and moment of inertia I. **(a)** Write down the total energy of the two masses and the pulley in terms of the coordinate x and $\dot{x}$. (Remember that the kinetic energy of a spinning wheel is $\frac{1}{2}I\omega^2$.) **(b)** Show (what is true for any conservative one-dimensional system) that you can obtain the equation of motion for the coordinate x by differentiating the equation $E = \text{const}$. Check that the equation of motion is the same as you would obtain by applying Newton's second law separately to the two masses and the pulley, and then eliminating the two unknown tensions from the three resulting equations.

(a) Because the string is inextensible, the height of m_2 below the wheel is $y = k - x$, where k is a constant, and its velocity is $\dot{y} = -\dot{x}$. The angular velocity of the wheel is $\omega = \dot{x}/R$. Thus the total energy is

$$E = T + U = \tfrac{1}{2}m_1\dot{x}^2 + \tfrac{1}{2}m_2\dot{x}^2 + \tfrac{1}{2}I\omega^2 - m_1 g x - m_2 g(k - x)$$
$$= \tfrac{1}{2}(m_1 + m_2 + I/R^2)\dot{x}^2 - (m_1 - m_2)gx,$$

if we drop the uninteresting constant $-m_2 g k$.
(b) The equation $dE/dt = 0$ yields $(m_1 + m_2 + I/R^2)\dot{x}\ddot{x} - (m_1 - m_2)g\dot{x} = 0$, or

$$(m_1 + m_2 + I/R^2)\ddot{x} = (m_1 - m_2)g. \tag{S4.5}$$

Applying the second law to each separate mass and to the rotating wheel, we find

$$m_1\ddot{x} = m_1 g - F_{\text{TR}}, \quad m_2\ddot{x} = F_{\text{TL}} - m_2 g, \quad \text{and} \quad I\dot{\omega} = (F_{\text{TR}} - F_{\text{TL}})R$$

where F_{TR} and F_{TL} are the tensions in the right and left lengths of string. Eliminating the two tensions, we get precisely Eq. (S4.5).

4.37 ⋆⋆⋆ [Computer] Figure 4.28 shows a massless wheel of radius R, mounted on a frictionless, horizontal axle. A point mass M is glued to the edge of the wheel, and a mass m hangs from a string wrapped around the perimeter of the wheel. **(a)** Write down the total PE of the two masses as a function of the angle ϕ. **(b)** Use this to find the values of m and M for which there are any positions of equilibrium. Describe the equilibrium positions, discuss their stability, and explain your answers in terms of torques. **(c)** Plot $U(\phi)$ for the cases that $m = 0.7M$ and $m = 0.8M$, and use your graphs to describe the behavior of the system if I release it from rest at $\phi = 0$. **(d)** Find the critical value of m/M on one side of which the system oscillates and on the other side of which it does not (if released from rest at $\phi = 0$).

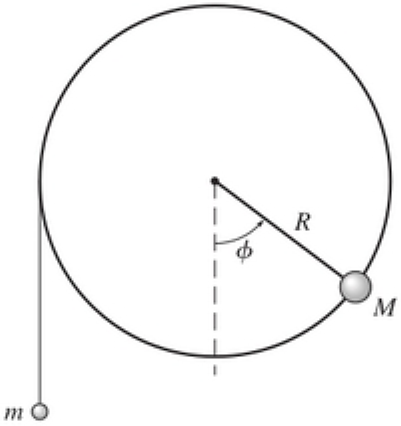

Figure 4.28 Problem 4.37

(a) Let us take the zero of PE when $\phi = 0$. As the wheel turns through angle ϕ, the mass M rises by $R(1 - \cos\phi)$ and m descends by $R\phi$. Therefore the total PE is

$$U(\phi) = MgR(1 - \cos\phi) - mgR\phi.$$

(b) The condition for equilibrium is that $dU/d\phi = 0$, that is,

$$MgR\sin\phi = mgR \quad \text{or} \quad \sin\phi = m/M. \tag{S4.6}$$

The equation $\sin\phi = m/M$ has no solutions if $m > M$. If $m = M$, there is one solution, at $\phi = \pi/2$. If $m < M$, there are two equilibrium angles, given by $\sin\phi = m/M$, one with M below the axle ($\phi < \pi/2$) and the other symmetrically located with M above the axle ($\phi > \pi/2$). All of these results are easy to understand in terms of torques. Equation (S4.6) is just the condition that the clockwise torque of M balance the counterclockwise torque of m. If $m > M$, they cannot balance; if $m < M$, they can balance in one position below the axle and one above.

The stability depends on the second derivative, $d^2U/d\phi^2 = MgR\cos\phi$. This is positive (stable equilibrium) for the equilibrium with $\phi < \pi/2$, but negative (unstable) for $\phi > \pi/2$. This also is easy to understand. If M is at the equilibrium below the axle and ϕ increases, the torque of M increases and returns M to its equilibrium. If M is at the equilibrium above the axle, an increase of ϕ reduces the torque of M, and M continues to move away from equilibrium. Both the stable and unstable equilibrium points are clearly visible as a valley and a hill in each of the pictures below. If $m = M$, the equilibrium is at $\phi = \pi/2$ and is unstable.

(c)

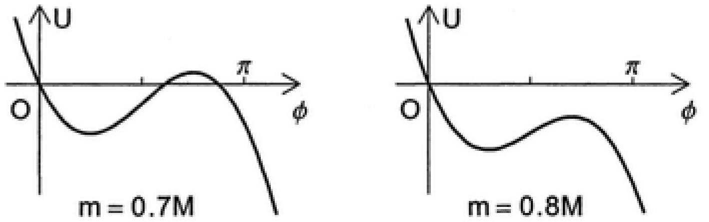

Figure S4.37c

The behavior of the wheel when released from $\phi = 0$ is quite different in the two cases $m = 0.7M$ or $0.8M$. When $m = 0.7M$, the hill that represents the unstable equilibrium is higher than zero; when the wheel is released from $\phi = 0$ it will rotate past the stable valley but will stop before it reaches the unstable hill, and swing back again. In this case the wheel oscillates indefinitely between the two turning points. When $m = 0.8M$, the unstable hill is lower than zero, and, when released from $\phi = 0$, the wheel will rotate past the stable valley *and on past the hill*. Once past the hill, ϕ will increase, and the wheel rotate counterclockwise indefinitely (or until the string holding m runs out).

(d) As the mass m increases, the tendency for our system to turn in the direction of increasing ϕ increases, and the graph of $U(\phi)$ against ϕ steadily tilts down to the right. (This is clearly visible in the two pictures for $m = 0.7M$ and $0.8M$.) Somewhere between these two cases there must be a critical value m_{crit} for which the unstable hill just touches the horizontal axis, as shown. To find when this happens notice that at this critical value, U vanishes at the same point as $dU/d\phi$. This gives two simultaneous equations:

$$M(1 - \cos\phi) = m\phi \quad [\text{that is, } U = 0]$$

and

$$M\sin\phi = m \quad [\text{that is, } dU/d\phi = 0]$$

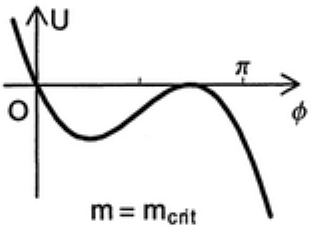

Figure S4.37d

By dividing one of these by the other, we can eliminate m and M to give an equation for ϕ, namely $1 - \cos\phi = \phi\sin\phi$. This equation cannot be solved analytically, but is easily solved with a numerical equation solver (such as FindRoot in Mathematica) to give $\phi = 2.33$ rad. From the second of the equations above, it follows that the critical value of the mass is $m = M\sin\phi = 0.725M$. The graph shown was drawn for this value of the mass and confirms that this is indeed the critical value. If m is any less, the wheel released from O will remain trapped and oscillate; if m is any more, the wheel will escape and rotate for ever.

4.38 ⋆⋆⋆ [Computer] Consider the simple pendulum of Problem 4.34. You can get an expression for the pendulum's period (good for large oscillations as well as small) using the method discussed in connection with (4.57), as follows: **(a)** Using (4.101) for the PE, find $\dot{\phi}$ as a function of ϕ. Next use (4.57), in the form $t = \int d\phi/\dot{\phi}$, to write the time for the pendulum to travel from $\phi = 0$ to its maximum value (the amplitude) Φ. Because this time is a quarter of the period τ, you can now write down the period. Show that

$$\tau = \tau_o \frac{1}{\pi}\int_0^{\Phi} \frac{d\phi}{\sqrt{\sin^2(\Phi/2) - \sin^2(\phi/2)}} = \tau_o \frac{2}{\pi}\int_0^1 \frac{du}{\sqrt{1-u^2}\sqrt{1-A^2u^2}}, \qquad (4.103)$$

where τ_o is the period (4.102) (Problem 4.34) for small oscillations and $A = \sin(\Phi/2)$. [To get the first expression you will need to use the trig identity for $1 - \cos\phi$ in terms of $\sin^2(\phi/2)$. To get the second you need to make the substitution $\sin(\phi/2) = Au$.] These integrals cannot be evaluated in terms of elementary functions. However, the second integral is a standard integral called the *complete elliptic* *integral of the first kind*, sometimes denoted $K(A^2)$, whose values are tabulated[2] and are known to computer software such as Mathematica [which calls it EllipticK(A^2)].

4.39 ⋆⋆⋆ **(a)** If you have not already done so, do Problem 4.38(a).
(b) If the amplitude Φ is small then so is $A = \sin(\Phi/2)$. If the amplitude is very small, we can simply ignore the last square root in (4.103). Show that this gives the familiar result for the small-amplitude period, $\tau = \tau_o = 2\pi\sqrt{l/g}$.
(c) If the amplitude is small but not very small, we can improve on the approximation of part (b). Use the binomial expansion to give the approximation $1/\sqrt{1 - A^2u^2} \approx 1 + \frac{1}{2}A^2u^2$ and show that, in this approximation, (4.103) gives

$$\tau = \tau_o[1 + \tfrac{1}{4}\sin^2(\Phi/2)].$$

What percentage correction does the second term represent for an amplitude of 45°? (The exact answer for $\Phi = 45°$ is $1.040\,\tau_o$ to four significant figures.)

(a) Using the trig identity $(1 - \cos\phi) = 2\sin^2(\phi/2)$, we can write the PE as $U(\phi) = 2mgl\sin^2(\phi/2)$. The KE is $\frac{1}{2}ml^2\dot{\phi}^2$ and the total energy is just $E = U(\Phi)$, since $T = 0$ at the endpoint $\phi = \Phi$. Thus we can solve the equation $T = U(\Phi) - U(\phi)$ to give

$$\dot{\phi} = \pm 2\sqrt{g/l}\sqrt{\sin^2(\Phi/2) - \sin^2(\phi/2)} = \pm\frac{4\pi}{\tau_o}\sqrt{\sin^2(\Phi/2) - \sin^2(\phi/2)}.$$

(To get the last expression I used $\sqrt{l/g} = \tau_o/2\pi$.) The time to swing from $\phi = 0$ to Φ is given by the integral $\int d\phi/\dot{\phi}$ taken from 0 to Φ (using the positive square root for $\dot{\phi}$). Since the period is four times this amount, we conclude that

$$\tau = 4\int_0^{\Phi}\frac{d\phi}{\dot{\phi}} = \frac{\tau_o}{\pi}\int_0^{\Phi}\frac{d\phi}{\sqrt{\sin^2(\Phi/2) - \sin^2(\phi/2)}}.$$

If we make the recommended substitutions, $\sin(\Phi/2) = A$ and $\sin(\phi/2) = Au$, the square root in the denominator becomes $A\sqrt{1 - u^2}$ and $d\phi = 2A\,du/\cos(\phi/2) = 2A\,du/\sqrt{1 - A^2u^2}$. Putting all this together, we find

$$\tau = \tau_o\frac{2}{\pi}\int_0^1\frac{du}{\sqrt{1 - u^2}\sqrt{1 - A^2u^2}} = \tau_o\frac{2}{\pi}K(A^2). \tag{S4.7}$$

(b) Ignoring the second square root completely, we get

$$\tau = \tau_o\frac{2}{\pi}\int_0^1\frac{du}{\sqrt{1 - u^2}} = \tau_o$$

as expected. (To show the integral is just $\pi/2$, use the substitution $u = \sin\alpha$.)
(c) If we make the binomial approximation suggested, we get

$$\tau = \tau_o\frac{2}{\pi}\left(\int_0^1\frac{du}{\sqrt{1 - u^2}} + \tfrac{1}{2}A^2\int_0^1\frac{u^2du}{\sqrt{1 - u^2}}\right).$$

The first integral is $\pi/2$ as before, while the same substitution shows the second integral to be $\pi/4$. Recalling that $A = \sin(\Phi/2)$, we find that $\tau = \tau_o[1 + \frac{1}{4}\sin^2(\Phi/2)]$, as stated. If $\Phi = 45°$, this gives

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[2] See, for example, M.Abramowitz and I.Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1965. Be warned that different authors use different notations. In particular, some authors call the exact same integral $K(A)$.

$\tau = 1.037\tau_o$, which represents a 3.7% correction to the small-amplitude approximation (τ_o), and is itself within 0.3% of the exact answer ($1.040\tau_o$).

SECTION 4.8 Central Forces

4.41 ⋆ A mass m moves in a circular orbit (centered on the origin) in the field of an attractive central force with potential energy $U = kr^n$. Prove the **virial theorem** that $T = nU/2$.

Since the mass moves in a circle, the radial component of its accelerations is just $-v^2/r$, the centripetal acceleration. By Newton's second law, $m(-v^2/r) = F_r = -dU/dr = -nkr^{n-1}$, from which we see that $mv^2 = nkr^n = nU$, and hence $T = \frac{1}{2}mv^2 = \frac{1}{2}nU$.

4.43 ⋆⋆ In Section 4.8, I claimed that a force $\mathbf{F}(\mathbf{r})$ that is central and spherically symmetric is automatically conservative. Here are two ways to prove it: **(a)** Since $\mathbf{F}(\mathbf{r})$ is central and spherically symmetric, it must have the form $\mathbf{F}(\mathbf{r}) = f(r)\hat{\mathbf{r}}$. Using Cartesian coordinates, show that this implies that $\nabla \times \mathbf{F} = 0$. **(b)** Even quicker, using the expression given inside the back cover for $\nabla \times \mathbf{F}$ in spherical polars, show that $\nabla \times \mathbf{F} = 0$.

(a) Since $\hat{\mathbf{r}} = \mathbf{r}/r$ we can write $\mathbf{F} = f(r)\mathbf{r}/r = g(r)\mathbf{r}$, say, where $g(r)$ is simply a convenient new name for $f(r)/r$. Now consider the x component of $\nabla \times \mathbf{F}$:

$$\begin{aligned}(\nabla \times \mathbf{F})_x &= \partial_y F_z - \partial_z F_y \\ &= \partial_y[g(r)z] - \partial_z[g(r)y] = z\,\partial_y g(r) - y\,\partial_z g(r).\end{aligned}$$

Now by the chain rule, $\partial_y g(r) = g'(r)\partial r/\partial y = g'(r)y/r$, and similarly for $\partial_z g(r)$. Therefore

$$(\nabla \times \mathbf{F})_x = zg'(r)y/r - yg'(r)z/r = 0.$$

Since the other two components work the same way, we conclude that $\nabla \times \mathbf{F} = 0$ and $\mathbf{F}(\mathbf{r})$ is conservative.

(b) If $\mathbf{F}(\mathbf{r})$ is central and spherically symmetric, its spherical polar components are $F_r = f(r)$ and $F_\theta = F_\phi = 0$. When we insert these in the given expression for $\nabla \times \mathbf{F}$ in spherical polars, only the two terms involving F_r survive since $F_\theta = F_\phi = 0$, but both of these remaining terms involve derivatives of F_r with respect to θ or ϕ, which are also zero. Therefore $\nabla \times \mathbf{F} = 0$.

4.45 ⋆⋆ In Section 4.8, I proved that a force $\mathbf{F}(\mathbf{r}) = f(\mathbf{r})\hat{\mathbf{r}}$ that is central and conservative is automatically spherically symmetric. Here is an alternative proof: Consider the two paths ACB and ADB of Figure 4.29, but with $r_B = r_A + dr$ where dr is infinitesimal. Write down the work done by $\mathbf{F}(\mathbf{r})$ going around both paths, and use the fact that they must be equal to prove that the magnitude function $f(\mathbf{r})$ must be the same at points A and D; that is, $f(\mathbf{r}) = f(r)$ and the force is spherically symmetric.

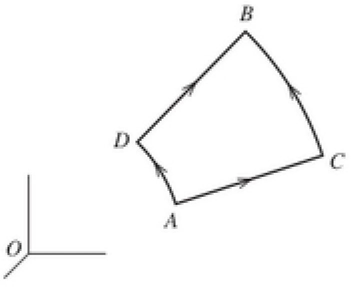

Figure 4.29 Problems 4.44 and 4.45

That $\mathbf{F}$ is conservative tells us that the amounts of work done along the two paths ACB and ADB are equal, $W_{ACB} = W_{ADB}$. That $\mathbf{F}$ is central implies that no work is done along CB and AD, that is, $W_{CB} = W_{AD} = 0$. Therefore $W_{AC} = W_{DB}$. Since $\mathbf{F} = f(\mathbf{r})\hat{\mathbf{r}}$, $W_{AC} = f(\mathbf{r}_A)dr$ and $W_{DB} = f(\mathbf{r}_D)dr$, and since these two are equal, it follows that $f(\mathbf{r}_A) = f(\mathbf{r}_D)$. Finally, since A and D are any two points at the same distance from O, this says that $f(\mathbf{r})$ depends only on $|\mathbf{r}|$. That is, $f(\mathbf{r}) = f(r)$.

SECTION 4.9 Energy of Interaction of Two Particles

4.47 ⋆ Consider a head-on elastic collision between two particles. (Since the collision is head-on, the motion is confined to a single straight line and is therefore one-dimensional.) Prove that the relative velocity after the collision is equal and opposite to that before. That is, $v_1 - v_2 = -(v_1' - v_2')$, where v_1 and v_2 are the initial velocities and v_1' and v_2' the corresponding final velocities.

Since the problem is one-dimensional, we can use v to denote velocity (strictly speaking the x component of velocity). Then conservation of KE (a little rearranged) says

$$m_1(v_1^2 - v_1'^2) = m_2(v_2'^2 - v_2^2)$$

and, similarly, conservation of momentum

$$m_1(v_1 - v_1') = m_2(v_2' - v_2).$$

Dividing the first of these by the second, we find

$$v_1 + v_1' = v_2' + v_2 \quad \text{or} \quad v_1 - v_2 = v_2' - v_1'.$$

4.49 ⋆⋆ Both the Coulomb and gravitational forces lead to potential energies of the form $U = \gamma/|\mathbf{r}_1 - \mathbf{r}_2|$, where γ denotes kq_1q_2 in the case of the Coulomb force and $-Gm_1m_2$ for gravity, and $\mathbf{r}_1$ and $\mathbf{r}_2$ are the positions of the two particles. Show in detail that $-\nabla_1 U$ is the force on particle 1 and $-\nabla_2 U$ that on particle 2.

Let's define $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, so that $\mathbf{r}$ is a vector pointing from particle 2 to particle 1. The PE is $U = \gamma/r$, the force on particle 1 due to particle 2 is $\mathbf{F}_{12} = (\gamma/r^2)\hat{\mathbf{r}}$, and that on particle 2 due to particle 1 is $\mathbf{F}_{21} = -(\gamma/r^2)\hat{\mathbf{r}}$. We'll start with the x component of $-\nabla_1 U$:

$$(-\nabla_1 U)_x = -\frac{\partial U}{\partial x_1} = \frac{\gamma}{r^2}\frac{\partial r}{\partial x_1} = \frac{\gamma}{r^3}(x_1 - x_2) \tag{S4.8}$$

where for the second equality I used the chain rule and for the third I used the following:

$$\frac{\partial r}{\partial x_1} = \frac{\partial}{\partial x_1}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} = \frac{x_1 - x_2}{\sqrt{\cdots}} = \frac{x_1 - x_2}{r}. \tag{S4.9}$$

Combining (S4.8) with the corresponding results for the y and z components, we see that

$$-\nabla_1 U = \frac{\gamma}{r^3}(\mathbf{r}_1 - \mathbf{r}_2) = \frac{\gamma}{r^3}\mathbf{r} = \frac{\gamma}{r^2}\hat{\mathbf{r}} = \mathbf{F}_{12}. \tag{S4.10}$$

If we had evaluated $-\nabla_2 U$, the only difference would have been that in the equivalent of (S4.9), we would have had an extra minus sign because $\partial(x_1 - x_2)^2/\partial x_2 = -2(x_1 - x_2)$. Thus in place of (S4.10), we would have found $-\nabla_2 U = -(\gamma/r^2)\hat{\mathbf{r}} = \mathbf{F}_{21}$.

SECTION 4.10 The Energy of a Multiparticle System

4.51 ⋆⋆ Write out the arguments of all the potential energies of the four-particle system in (4.94). For instance $U = U(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_4)$, whereas $U_{34} = U_{34}(\mathbf{r}_3 - \mathbf{r}_4)$. Show in detail that the net force on particle 3 (for instance) is given by $-\nabla_3 U$. [You know that the separate forces, internal and external, are given by (4.92) and (4.93).]

$$\begin{aligned} U(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = U_{12}(\mathbf{r}_1 - \mathbf{r}_2) &+ U_{13}(\mathbf{r}_1 - \mathbf{r}_3) + U_{14}(\mathbf{r}_1 - \mathbf{r}_4) \\ &+ U_{23}(\mathbf{r}_2 - \mathbf{r}_3) + U_{24}(\mathbf{r}_2 - \mathbf{r}_4) + U_{34}(\mathbf{r}_3 - \mathbf{r}_4) \\ &+ U_1^{\text{ext}}(\mathbf{r}_1) + U_2^{\text{ext}}(\mathbf{r}_2) + U_3^{\text{ext}}(\mathbf{r}_3) + U_4^{\text{ext}}(\mathbf{r}_4) \end{aligned}$$

so

$$\begin{aligned} -\nabla_3 U = 0 &+ \mathbf{F}_{31} + 0 \\ &+ \mathbf{F}_{32} + 0 + \mathbf{F}_{34} \\ &+ 0 + 0 + \mathbf{F}_3^{\text{ext}} + 0 = \mathbf{F}_3^{\text{net}}. \end{aligned}$$

4.53 ⋆⋆ **(a)** Consider an electron (charge $-e$ and mass m) in a circular orbit of radius r around a fixed proton (charge $+e$). Remembering that the inward Coulomb force ke^2/r^2 is what gives the electron its centripetal acceleration, prove that the electron's KE is equal to $-\frac{1}{2}$ times its PE; that is, $T = -\frac{1}{2}U$ and hence $E = \frac{1}{2}U$. (This result is a consequence of the so-called *virial theorem*. See Problem 4.41.) Now consider the following inelastic collision of an electron with a hydrogen atom: Electron number 1 is in a circular orbit of radius r around a fixed proton. (This is the hydrogen atom.) Electron 2 approaches from afar with kinetic energy T_2. When the second electron hits the atom, the first electron is knocked free, and the second is captured in a circular orbit of radius r'. **(b)** Write down an expression for the total energy of the three-particle system in general. (Your answer should contain five terms, three PEs but only two KEs, since the proton is considered fixed.) **(c)** Identify the values of all five terms and the total energy E long before the collision occurs, and again long after it is all over. What is the KE of the outgoing electron 1 once it is far away? Give your answers in terms of the variables T_2, r, and r'.

(a) The centripetal acceleration is supplied by the Coulomb force, so $mv^2/r = ke^2/r^2$ and thence $T = \frac{1}{2}mv^2 = \frac{1}{2}ke^2/r = -\frac{1}{2}U$.

(b) $E = T_1 + T_2 + U_1 + U_2 + U_{12} = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 - ke^2\left(\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{r_{12}}\right)$.

(c) Long before: $E = T_1 + T_2 + U_1 + 0 + 0 = T_2 - \dfrac{ke^2}{2r}$.

Long after: $E' = T_1' + T_2' + 0 + U_2' + 0 = T_1' - \dfrac{ke^2}{2r'}$.

Therefore, by conservation of energy, $T_1' = T_2 + \frac{1}{2}ke^2\left(\frac{1}{r'} - \frac{1}{r}\right)$.

CHAPTER 5

Oscillations

PROBLEMS AND SOLUTIONS

SECTION 5.1 Hooke's Law

5.1 ⋆ A massless spring has unstretched length l_o and force constant k. One end is now attached to the ceiling and a mass m is hung from the other. The equilibrium length of the spring is now l_1. **(a)** Write down the condition that determines l_1. Suppose now the spring is stretched a further distance x beyond its new equilibrium length. Show that the net force (spring plus gravity) on the mass is $F = -kx$. That is, the net force obeys Hooke's law, when x is the distance from the equilibrium position — a very useful result, which lets us treat a mass on a vertical spring just as if it were horizontal. **(b)** Prove the same result by showing that the net potential energy (spring plus gravity) has the form $U(x) = \text{const} + \frac{1}{2}kx^2$.

(a) When the spring is at the new equilibrium position it is extended by an amount $l_1 - l_o$, so its tension is $k(l_1 - l_o)$. This must balance the weight mg. Therefore, $k(l_1 - l_o) = mg$. When it is stretched a further x, its total extension is $x + l_1 - l_o$ and the tension is $k(x + l_1 - l_o)$ upward. Thus the total downward force on the mass is

$$F = -k(x + l_1 - l_o) + mg = -kx.$$

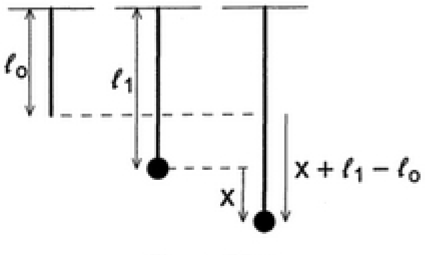

Figure S5.1

(b) The total PE is $U = U_{\text{sp}} + U_{\text{gr}}$ or

$$U = \tfrac{1}{2}k(x + l_1 - l_o)^2 - mgx = \tfrac{1}{2}kx^2 + k(l_1 - l_o)x - mgx + \tfrac{1}{2}k(l_1 - l_o)^2 = \tfrac{1}{2}kx^2 + \text{const}$$

5.3 ⋆ Write down the potential energy $U(\phi)$ of a simple pendulum (mass m, length l) in terms of the angle ϕ between the pendulum and the vertical. (Choose the zero of U at the bottom.) Show that, for small angles, U has the Hooke's law form $U(\phi) = \frac{1}{2}k\phi^2$, in terms of the coordinate ϕ. What is k?

The height of the mass below the pivot is $l\cos\phi$. Therefore the height above the bottom is $l(1-\cos\phi)$ and the PE is $U = mgl(1-\cos\phi)$. If ϕ is small, $\cos\phi \approx 1 - \frac{1}{2}\phi^2$ and $U \approx \frac{1}{2}mgl\phi^2 = \frac{1}{2}k\phi^2$, where $k = mgl$.

SECTION 5.2 Simple Harmonic Motion

5.5 ⋆ In Section 5.2 we discussed four equivalent ways to represent simple harmonic motion in one dimension:

$$\begin{aligned} x(t) &= C_1 e^{i\omega t} + C_2 e^{-i\omega t} && \text{(I)} \\ &= B_1\cos(\omega t) + B_2\sin(\omega t) && \text{(II)} \\ &= A\cos(\omega t - \delta) && \text{(III)} \\ &= \operatorname{Re} C e^{i\omega t} && \text{(IV)} \end{aligned}$$

To make sure you understand all of these, show that they are equivalent by proving the following implications: I ⇒ II ⇒ III ⇒ IV ⇒ I. For each form, give an expression for the constants (C_1, C_2, etc.) in terms of the constants of the previous form.

(a) In the form (I), $x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$, we can replace the exponentials using Euler's formula $e^{\pm i\omega t} = \cos\omega t \pm i\sin\omega t$ to give

$$x(t) = B_1\cos\omega t + B_2\sin\omega t \qquad \text{(II)}$$

where $B_1 = C_1 + C_2$ and $B_2 = i(C_1 - C_2)$.
(b) To get form (III) from form (II), define A and δ to be the hypotenuse and lower angle of a right triangle with base B_1 and height B_2 as in Figure 5.4. Then (II) can be rewritten as in Equation (5.11) to give the form (III).
(c) Since $\cos\theta = \operatorname{Re} e^{i\theta}$, we can rewrite the form (III) as

$$x(t) = \operatorname{Re} A e^{i(\omega t - \delta)} = \operatorname{Re} C e^{i\omega t} \qquad \text{(IV)}$$

where $C = Ae^{-i\delta}$.
(d) Finally, since $\operatorname{Re} z = \frac{1}{2}(z + z^*)$, we can rewrite (IV) as

$$x(t) = \tfrac{1}{2}(Ce^{i\omega t} + C^* e^{-i\omega t}) = C_1 e^{i\omega t} + C_2 e^{-i\omega t} \qquad \text{(I)}$$

where $C_1 = C/2$ and $C_2 = C^*/2$.

5.7 ⋆ **(a)** Solve for the coefficients B_1 and B_2 of the form (II) of Problem 5.5 in terms of the initial position x_o and velocity v_o at $t = 0$. **(b)** If the oscillator's mass is $m = 0.5$ kg and the force constant is $k = 50$ N/m, what is the angular frequency ω? If $x_o = 3.0$ m and $v_o = 50$ m/s, what are B_1 and B_2? Sketch $x(t)$ for a couple of cycles. **(c)** What are the earliest times at which $x = 0$ and at which $\dot{x} = 0$?

(a) Since $x(t) = B_1\cos\omega t + B_2\sin\omega t$, $x(0) = B_1$ and $v(0) = \omega B_2$. Therefore, $B_1 = x_o$ and $B_2 = v_o/\omega$.
(b) If $k = 50$ N/m and $m = 0.5$ kg, then $\omega = \sqrt{k/m} = 10\ \text{s}^{-1}$. If $x_o = 3.0$ m and $v_o = 50$ m/s, then $B_1 = 3$ m, $B_2 = 5$ m.

(c) If we rewrite $x(t)$ in the form $x(t) = A\cos(\omega t - \delta)$, with $\delta = \arctan(B_2/B_1) = 1.03$ rad, then x first vanishes when $(\omega t - \delta) = \pi/2$ or $t = 0.26$ s, and v first vanishes when $(\omega t - \delta) = 0$ or $t = \delta/\omega = 0.10$ s.

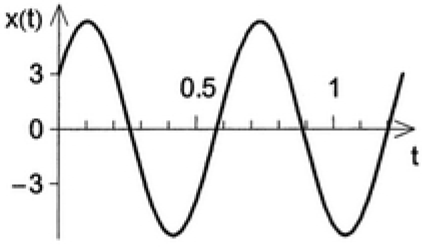

Figure S5.7

5.9 ⋆ The maximum displacement of a mass oscillating about its equilibrium position is 0.2 m, and its maximum speed is 1.2 m/s. What is the period τ of its oscillations?

We are given that $A = 0.2$ m and $v_o = 1.2$ m/s. We know that $E = \frac{1}{2}kA^2 = \frac{1}{2}mv_o^2$. Therefore $m/k = A^2/v_o^2$ and $\tau = 2\pi\sqrt{m/k} = 2\pi A/v_o = 1.05$ s.

5.11 ⋆ You are told that, at the known positions x_1 and x_2, an oscillating mass m has speeds v_1 and v_2. What are the amplitude and the angular frequency of the oscillations?

The given information gives two expressions of the total energy E

$$E = \tfrac{1}{2}mv_1^2 + \tfrac{1}{2}kx_1^2 \qquad \text{and} \qquad E = \tfrac{1}{2}mv_2^2 + \tfrac{1}{2}kx_2^2. \tag{S5.1}$$

Equating these two, we find $m(v_1^2 - v_2^2) = k(x_2^2 - x_1^2)$. This implies that

$$\omega^2 = \frac{k}{m} = \frac{v_1^2 - v_2^2}{x_2^2 - x_1^2}.$$

We know also that $E = \frac{1}{2}kA^2$, and inserting this in the first of Eqs. (S5.1) we conclude that

$$A^2 = \frac{m}{k}v_1^2 + x_1^2 = \frac{x_2^2 - x_1^2}{v_1^2 - v_2^2}v_1^2 + x_1^2 = \frac{x_2^2 v_1^2 - x_1^2 v_2^2}{v_1^2 - v_2^2}.$$

5.13 ⋆⋆ The potential energy of a one-dimensional mass m at a distance r from the origin is

$$U(r) = U_o\left(\frac{r}{R} + \lambda^2\frac{R}{r}\right)$$

for $0 < r < \infty$, with U_o, R, and λ all positive constants. Find the equilibrium position r_o. Let x be the distance from equilibrium and show that, for small x, the PE has the form $U = \text{const} + \frac{1}{2}kx^2$. What is the angular frequency of small oscillations?

Since $U(r) = U_o(r/R + \lambda^2 R/r)$, its derivative is $dU/dr = U_o(1/R - \lambda^2 R/r^2)$, which vanishes at $r = \lambda R$ and nowhere else. Clearly $U(r) \to +\infty$ when $r \to 0$ or ∞, so $U(r)$ has a minimum at $r = r_o = \lambda R$. If we write $r = r_o + x$ then

$$U = \lambda U_o\left(\frac{r_o+x}{r_o}+\frac{r_o}{r_o+x}\right) = \lambda U_o\left(1+\frac{x}{r_o}+\left[1+\frac{x}{r_o}\right]^{-1}\right)$$

$$\approx \lambda U_o\left(1+\frac{x}{r_o}+1-\frac{x}{r_o}+\frac{x^2}{r_o^2}\right) = \lambda U_o\left(2+\frac{x^2}{\lambda^2 R^2}\right).$$

where, in the second line, I dropped all terms in $(x/r_o)^3$ and higher. This has the expected form $U = \frac{1}{2}kx^2 + \text{const}$, where $k = 2U_o/(\lambda R^2)$. The angular frequency is $\omega = \sqrt{k/m} = \sqrt{2U_o/(m\lambda R^2)}$.

SECTION 5.3 Two-Dimensional Oscillators

5.15 ⋆ The general solution for a two-dimensional isotropic oscillator is given by (5.19). Show that by changing the origin of time you can cast this in the simpler form (5.20) with $\delta = \delta_y - \delta_x$. [*Hint:* A change of origin of time is a change of variables from t to $t' = t + t_o$. Make this change and choose the constant t_o appropriately, then rename t' to be t.]

If we replace the variable t by $t' = t + t_o$, Eq. (5.19) becomes

$$x = A_x\cos(\omega t' - \omega t_o - \delta_x) \quad \text{and} \quad y = A_y\cos(\omega t' - \omega t_o - \delta_y)$$

and if we then choose t_o such that $\omega t_o = -\delta_x$ these become

$$x = A_x\cos(\omega t') \quad \textit{and} \quad y = A_y\cos(\omega t' - [\delta_y - \delta_x]).$$

If we rename t' as t and set $\delta_y - \delta_x = \delta$, this is the desired form.

5.17 ⋆⋆ Consider the two-dimensional anisotropic oscillator with motion given by Equation (5.23). **(a)** Prove that if the ratio of frequencies is rational (that is, $\omega_x/\omega_y = p/q$ where p and q are integers) then the motion is periodic. What is the period? **(b)** Prove that if the same ratio is irrational, the motion never repeats itself.

(a) Suppose that the ratio of frequencies is rational, that is $\omega_x/\omega_y = p/q$, where p and q are integers. Then let $\tau = 2\pi p/\omega_x = 2\pi q/\omega_y$. Now consider the following

$$x(t+\tau) = A_x\cos[\omega_x(t+\tau)] = A_x\cos[\omega_x t + 2\pi p] = A_x\cos[\omega_x t] = x(t)$$

where in the second equality I used our definition of τ and in the third the fact that if p is an integer then $\cos(\theta + 2\pi p) = \cos(\theta)$. This shows that $x(t)$ is periodic with period τ. By exactly the same argument, $y(t)$ is also periodic with the same period τ, and we've proved that the whole motion is likewise. What we usually call *the* period of the motion is the value of $\tau = 2\pi p/\omega_x$ with p and q the *smallest* integers for which $\omega_x/\omega_y = p/q$.
(b) Suppose the motion is periodic. Then there is a τ such that $x(t+\tau) = x(t)$ and $y(t+\tau) = y(t)$. Running the previous argument backward, we see that $\omega_x\tau$ must be an integer multiple of 2π, that is $\omega_x\tau = 2\pi p$ for some integer p. Similarly $\omega_y\tau = 2\pi q$ for some integer q. Dividing these two conclusions, we see that $\omega_x/\omega_y = p/q$ and the ratio of frequencies is rational. Therefore, if the ratio is irrational, the motion cannot be periodic.

5.18 ⋆⋆⋆ The mass shown in Figure 5.27 is resting on a frictionless horizontal table. Each of the two identical springs has force constant k and unstretched length l_o. At equilibrium the mass rests at the origin, and the distances a are not necessarily equal to l_o. (That is, the springs may already be stretched or compressed.) Show that when the mass moves to a position (x, y), with x and y small, the potential energy has the form (5.104) (Problem 5.14) for an anisotropic oscillator. Show that if $a < l_o$ the equilibrium at the origin is unstable and explain why.

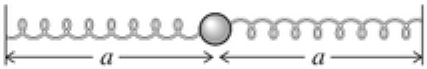

Figure 5.27 Problem 5.18

When the mass is at position (x, y), the lengths of the two springs are l_1 and l_2, where

$$l_1 = \sqrt{(a+x)^2 + y^2} = a\left(1 + \frac{2x}{a} + \frac{x^2+y^2}{a^2}\right)^{1/2}$$
$$\approx a\left[1 + \frac{1}{2}\left(\frac{2x}{a} + \frac{x^2+y^2}{a^2}\right) - \frac{1}{8}\left(\frac{2x}{a}\right)^2\right] = a + x + \frac{y^2}{2a}.$$

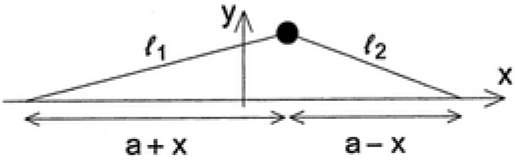

Figure S5.18

Here, in passing from the first to the second line, I have used the Taylor expansion $(1+\epsilon)^{1/2} = 1 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \cdots$, dropping all terms of third degree in x or y, but being careful to keep all terms of second degree. The PE of spring 1 is therefore

$$U_1 = \tfrac{1}{2}k(l_1 - l_o)^2 = \tfrac{1}{2}k[(a - l_o) + x + y^2/2a]^2$$
$$\approx \tfrac{1}{2}k[(a - l_o)^2 + 2(a - l_o)x + x^2 + (1 - l_o/a)y^2]$$

where, again, I have dropped terms of degree three in x or y. To find U_2, we have only to replace x by $-x$, and for the total PE we just add U_1 and U_2. When we do this, the terms linear in x cancel, leaving

$$U = U_1 + U_2 = k[x^2 + (1 - l_o/a)y^2] + \text{const}$$

which has the form (5.104), apart from the unimportant constant.

If $a < l_o$, the coefficient of y^2 is negative and the equilibrium at the origin O is unstable. This is because, with $a < l_o$, the springs are in compression at O. When the mass moves a little from O along the y axis, the compression in the springs forces it further away, causing the instability.

5.19 ⋆⋆⋆ Consider the mass attached to four identical springs, as shown in Figure 5.7(b). Each spring has force constant k and unstretched length l_o, and the length of each spring when the mass is at its equilibrium at the origin is a (not necessarily the same as l_o). When the mass is displaced a small distance to the point (x, y), show that its potential energy has the form $\frac{1}{2}k'r^2$ appropriate to an isotropic harmonic oscillator. What is the constant k' in terms of k? Give an expression for the corresponding force.

The simplest way to find the total PE of all four springs is treat them two at a time. The two springs anchored on the x axis constitute the system of Problem 5.18, for which we found that

$$U_1 + U_2 = k[x^2 + (1 - l_o/a)y^2].$$

(See the solution to Problem 5.18, included above. I've dropped an uninteresting constant here.) In exactly the same way, the PE of the two springs anchored on the y axis is

$$U_3 + U_4 = k[y^2 + (1 - l_o/a)x^2].$$

Adding these, we find for the total PE of all four springs

$$U = U_1 + U_2 + U_3 + U_4 = k[x^2 + y^2 + (1 - l_o/a)(x^2 + y^2)] = k\frac{2a - l_o}{a}r^2,$$

where I've used the fact that $x^2 + y^2 = r^2$. This has the advertised form $U = \frac{1}{2}k'r^2$ with an effective spring constant $k' = 2k(2a - l_o)/a$. The corresponding force is $\mathbf{F} = -\nabla U = -k'\mathbf{r}$.

SECTION 5.4 Damped Oscillations

5.21 ⋆ Verify that the function (5.43), $x(t) = te^{-\beta t}$, is indeed a second solution of the equation of motion (5.28) for a critically damped oscillator ($\beta = \omega_o$).

If $x = te^{-\beta t}$, then

$$\dot{x} = (1 - \beta t)e^{-\beta t} \qquad \text{and} \qquad \ddot{x} = (\beta^2 t - 2\beta)e^{-\beta t}.$$

Therefore

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = \left[(\beta^2 t - 2\beta) + 2\beta(1 - \beta t) + \omega_o^2 t\right]e^{-\beta t} = [\omega_o^2 - \beta^2]e^{-\beta t} = 0$$

if (and only if) $\beta = \omega_o$. That is, $x = te^{-\beta t}$ *does* satisfy Eq. (5.28) if the damping is critical.

5.23 ⋆ A damped oscillator satisfies the equation (5.24), where $F_{\text{dmp}} = -b\dot{x}$ is the damping force. Find the rate of change of the energy $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ (by straightforward differentiation), and, with the help of (5.24), show that dE/dt is (minus) the rate at which energy is dissipated by F_{dmp}.

Because $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$,

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + kx\dot{x} = \dot{x}(m\ddot{x} + kx) = \dot{x}(-b\dot{x}) = vF_{\text{dmp}}$$

where, for the third equality, I used the differential equation (5.24). Since vF_{dmp} is the rate at which F_{dmp} does work on the oscillator, this is the requested result.

5.25 ⋆⋆ Consider a damped oscillator with $\beta < \omega_o$. There is a little difficulty defining the "period" τ_1 since the motion (5.38) is not periodic. However, a definition that makes sense is that τ_1 is the time between successive maxima of $x(t)$. **(a)** Make a sketch of $x(t)$ against t and indicate this definition of τ_1 on your graph. Show that $\tau_1 = 2\pi/\omega_1$. **(b)** Show that an equivalent definition is that τ_1 is twice the time between successive zeros of $x(t)$. Show this one on your sketch. **(c)** If $\beta = \omega_o/2$, by what factor does the amplitude shrink in one period?

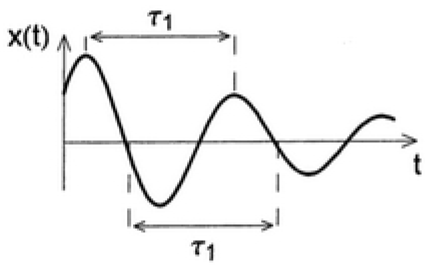

Figure S5.25

(a) Because $x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)$, its derivative is $dx/dt = -Ae^{-\beta t}[\beta\cos(\cdots) + \omega_1\sin(\cdots)]$. The maxima and minima of $x(t)$ occur when this derivative vanishes, that is, when $\tan(\omega_1 t - \delta) = -\beta/\omega_1$. Because $\tan\theta$ is π-periodic, the zeroes of dx/dt are equally spaced, with separation π/ω_1. The zeroes of the derivatives correspond alternately to maxima and minima, so the maxima are separated by a time $\tau_1 = 2\pi/\omega_1$.
(b) The zeroes of $x(t)$ occur when $\cos(\omega_1 t - \delta) = 0$. Thus they are regularly spaced with separation of π/ω_1, which equals $\tau_1/2$.
(c) With $\beta = \omega_o/2$, the amplitude shrinks by a factor

$$e^{-\beta\tau_1} = e^{-2\pi\beta/\sqrt{\omega_o^2-\beta^2}} = e^{-2\pi/\sqrt{3}} = 0.027$$

(This is much more shrinkage than in the picture, for which β was chosen to be $\omega_o/10$ and the shrinkage factor is about 0.53).

5.27 ⋆⋆ As the damping on an oscillator is increased there comes a point when the name "oscillator" seems barely appropriate. **(a)** To illustrate this, prove that a critically damped oscillator can never pass through the origin $x = 0$ more than once. **(b)** Prove the same for an overdamped oscillator.

The question is: How many times can the function $x(t)$ vanish? If the oscillator is weakly damped ($\beta < \omega_o$), then according to Eq. (5.38) $x(t)$ contains a factor $\cos(\omega t - \delta)$, which vanishes infinitely many times as it oscillates.
(a) If the oscillator is critically damped ($\beta = \omega_o$), then, according to (5.44), $x(t) = e^{-\beta t}(C_1 + C_2 t)$. This vanishes if and only if $t = -C_1/C_2$; therefore, $x(t)$ vanishes at most once. (It may never vanish—for example, if the motion starts at $t = 0$ and $-C_1/C_2 < 0$.)
(b) If the oscillator is overdamped ($\beta > \omega_o$), then according to (5.40),

$$x(t) = e^{-\beta t}(C_1 e^{\lambda t} + C_2 e^{-\lambda t}) = e^{-(\beta-\lambda)t}(C_1 + C_2 e^{-2\lambda t})$$

where $\lambda = \sqrt{\beta^2 - \omega_o^2}$. This vanishes if and only if $e^{-2\lambda t} = -C_1/C_2$, and, since $e^{-2\lambda t}$ is a monotonic function (always decreasing), this happens at most once.

5.29 ⋆⋆ An undamped oscillator has period $\tau_o = 1$ second. When weak damping is added, it is found that the amplitude of oscillation drops by 50% in one period τ_1. (The period of the damped oscillations is defined as the time between successive maxima, $\tau_1 = 2\pi/\omega_1$. See Problem 5.25.) How big is β compared to ω_o? What is τ_1?

After one period, the amplitude has shrunk by a factor $e^{-\beta\tau_1} = 0.5$. Therefore $\beta\tau_1 = \ln 2$. Meanwhile, because $\tau_1 = 2\pi/\omega_1 = 2\pi/\sqrt{\omega_o^2 - \beta^2}$, it follows that $\beta\tau_1 = 2\pi/\sqrt{(\omega_o/\beta)^2 - 1}$. Equating these two expressions for $\beta\tau_1$ and solving, we find $\beta/\omega_o = 1/\sqrt{1 + (2\pi/\ln 2)^2} = 0.1097$. That is, $\beta = 0.11\omega_o$.

$$\frac{\tau_1}{\tau_o} = \frac{\omega_o}{\omega_1} = \frac{\omega_o}{\sqrt{\omega_o^2 - \beta^2}} = \frac{1}{\sqrt{1 - \beta^2/\omega_o^2}} \approx 1 + \frac{1}{2}\frac{\beta^2}{\omega_o^2} = 1.006.$$

Therefore $\tau_1 = 1.006$ s.

5.31 ⋆⋆ [Computer] Consider a cart on a spring with natural frequency $\omega_o = 2\pi$, which is released from rest at $x_o = 1$ and $t = 0$. Using appropriate graphing software, plot the position $x(t)$ for $0 < t < 2$ and for damping constants $\beta = 0, 1, 2, 4, 6, 2\pi, 10$, and 20. [Remember that $x(t)$ is given by different formulas for $\beta < \omega_o$, $\beta = \omega_o$, and $\beta > \omega_o$.]

Starting with $\beta < \omega_o$, we know that $x(t)$ has the form (5.37), which we can rewrite as $x(t) = e^{-\beta t}(A\cos\omega_1 t + B\sin\omega_1 t)$, where, as usual, $\omega_1 = \sqrt{\omega_o^2 - \beta^2}$. Given that $x(0) = 1$ and $\dot{x}(0) = 0$, we can solve for A and B and find

$$x(t) = e^{-\beta t}\left(\cos(\omega_1 t) + \frac{\beta}{\omega_1}\sin(\omega_1 t)\right). \tag{S5.2}$$

Given that $\omega_o = 2\pi$, we can use this to plot $x(t)$ for any value of $\beta < \omega_o$. You can also use it for $\beta > \omega_o$. The only complication is that ω_1 is now imaginary, $\omega_1 = i\sqrt{\beta^2 - \omega_o^2} = i\tilde{\omega}_1$, say. If your plotting software is happy to work with complex numbers, you can continue to use exactly the form (S5.2), but it may be safer to rewrite it in terms of real functions as

$$x(t) = e^{-\beta t}\left(\cosh(\tilde{\omega}_1 t) + \frac{\beta}{\tilde{\omega}_1}\sinh(\tilde{\omega}_1 t)\right). \tag{S5.3}$$

Finally for $\beta = \omega_o$, you almost certainly need to use (5.44) which, for the given initial conditions has the form $x(t) = e^{-\beta t}(1 + \beta t)$. The results are shown below.

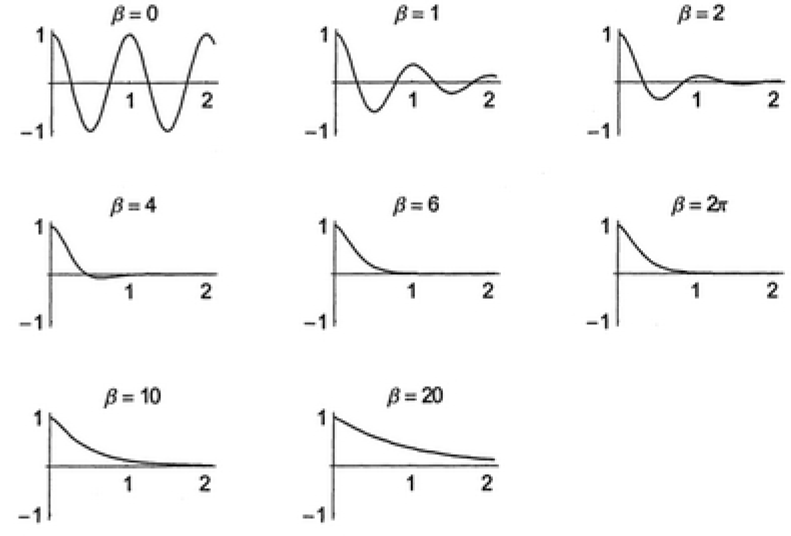

Figure S5.31

SECTION 5.5 Driven Damped Oscillations

5.33 ⋆ The solution for $x(t)$ for a driven, underdamped oscillator is most conveniently found in the form (5.69). Solve that equation and the corresponding expression for $\dot{x}$, to give the coefficients B_1 and B_2 in terms of A, δ, and the initial position and velocity x_o and v_o. Verify the expressions given in (5.70).

According to Eq. (5.69), $x(t) = A\cos(\omega t - \delta) + e^{-\beta t}[B_1\cos(\omega_1 t) + B_2\sin(\omega_1 t)]$. Setting $t = 0$, we obtain

$$x_o = x(0) = A\cos\delta + B_1 \quad\Longrightarrow\quad B_1 = x_o - A\cos\delta.$$

Similarly, differentiating $x(t)$ and then setting $t = 0$, we find

$$v_o = \dot{x}(0) = A\omega\sin\delta - \beta B_1 + \omega_1 B_2 \quad\Longrightarrow\quad B_2 = \frac{1}{\omega_1}(v_o - A\omega\sin\delta + \beta B_1)$$

as in Eq. (5.70).

5.35 ⋆⋆ This problem is to refresh your memory about some properties of complex numbers needed at several points in this chapter, but especially in deriving the resonance formula (5.64). **(a)** Prove that any complex number $z = x + iy$ (with x and y real) can be written as $z = re^{i\theta}$ where r and θ are the polar coordinates of z in the complex plane. (Remember Euler's formula.) **(b)** Prove that the absolute value of z, defined as $|z| = r$, is also given by $|z|^2 = zz^*$, where z^* denotes the *complex conjugate* of z, defined as $z^* = x - iy$. **(c)** Prove that $z^* = re^{-i\theta}$. **(d)** Prove that $(zw)^* = z^*w^*$ and that $(1/z)^* = 1/z^*$. **(e)** Deduce that if $z = a/(b + ic)$, with a, b, and c real, then $|z|^2 = a^2/(b^2 + c^2)$.

(a) $z = x + iy = r\cos\theta + i(r\sin\theta) = r(\cos\theta + i\sin\theta) = re^{i\theta}$.
(b) $zz^* = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$.
(c) $z^* = x - iy = r\cos\theta - i(r\sin\theta) = r[\cos(-\theta) + i\sin(-\theta)] = re^{-i\theta}$
(d) If $z = re^{i\theta}$ and $w = se^{i\phi}$, then $zw = rse^{i(\theta+\phi)}$. Therefore, $(zw)^* = rse^{-i(\theta+\phi)} = (re^{-i\theta})(se^{-i\phi}) = z^*w^*$. If $z = re^{i\theta}$, then $1/z = 1/(re^{i\theta}) = (1/r)e^{-i\theta}$, so $(1/z)^* = (1/r)e^{i\theta}$. Finally, $1/z^* = 1/(re^{-i\theta}) = (1/r)e^{i\theta} = (1/z)^*$.
(e) If $z = \dfrac{a}{b+ic}$, then $|z|^2 = zz^* = \dfrac{a}{b+ic}\left(\dfrac{a}{b+ic}\right)^* = \dfrac{a}{b+ic}\cdot\dfrac{a}{b-ic} = \dfrac{a^2}{b^2+c^2}$.

5.37 ⋆⋆ [Computer] Repeat the calculations of Example 5.3 (page 185) but with the following parameters

$$\omega = 2\pi, \qquad \omega_o = 0.25\omega, \qquad \beta = 0.2\omega_o, \qquad f_o = 1000$$

and with the initial conditions $x_o = 0$ and $v_o = 0$. Plot $x(t)$ for $0 \le t \le 12$ and compare with the plot of Example 5.3. Explain the similarities and differences. (It will help your explanation if you plot the homogeneous solution as well as the complete solution — homogeneous plus particular.)

With the help of Eqs. (5.64), (5.65), and (5.70) we find that $A = 26.9$, $\delta = 3.04$, $B_1 = 26.7$, and $B_2 = -6.18$. The corresponding motion is shown by the solid curve, while the dashed curve is the transient alone.

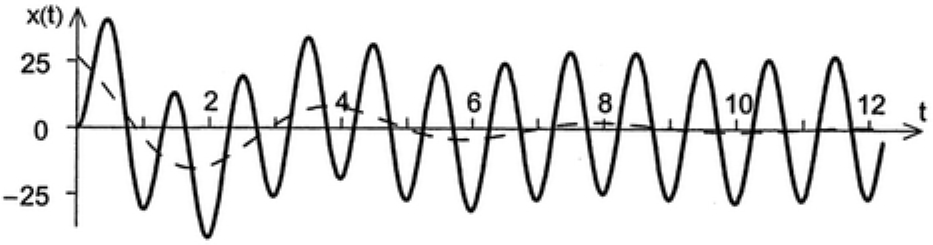

Figure S5.37

The most obvious difference between the present case and that of Example 5.3 is that here $\omega_o < \omega$, so that the transient imposes a slow rise and fall on the driven oscillations. In Fig.5.15 of Example 5.3, where $\omega_o > \omega$, the transient superposes a rapid oscillation on the driven motion. A subtler difference is that in Fig.5.15 where $\omega_o > \omega$ the phase shift δ is close to zero and the motion is in step with the driving force. Here, with $\omega_o < \omega$, $\delta \approx \pi$ and the motion is nearly 180° out of phase with the driving force. [The driving force has maxima at integer values of t whereas $x(t)$ has minima.]

5.39 ⋆⋆ [Computer] To get some practice at solving differential equations numerically, repeat the calculations of Example 5.3 (page 185), but instead of finding all the various coefficients just use appropriate software (for example, the NDSolve command of Mathematica) to solve the differential equation (5.48) with the boundary conditions $x_o = v_o = 0$. Make sure your graph agrees with Figure 5.15.

See Figure 5.15.

SECTION 5.6 Resonance

5.41 ⋆ We know that if the driving frequency ω is varied, the maximum response (A^2) of a driven damped oscillator occurs at $\omega \approx \omega_o$ (if the natural frequency is ω_o and the damping constant $\beta \ll \omega_o$). Show that A^2 is equal to half its maximum value when $\omega \approx \omega_o \pm \beta$, so that the full width at half maximum is just 2β. [*Hint:* Be careful with your approximations. For instance, it's fine to say $\omega + \omega_o \approx 2\omega_o$, but you certainly mustn't say $\omega - \omega_o \approx 0$.]

Provided β is significantly less that ω_o, the maximum of A^2 comes when $\omega \approx \omega_o$ and, at this point, the denominator of Eq. (5.71) is approximately $4\beta^2\omega_o^2$. Thus A^2 is equal to half its maximum when the denominator is equal to $8\beta^2\omega_o^2$, or when $(\omega^2 - \omega_o^2)^2 = 4\beta^2\omega_o^2$. This simplifies to $(\omega - \omega_o)(\omega + \omega_o) = \pm 2\beta\omega_o$. Since $(\omega + \omega_o) \approx 2\omega_o$, this says that the half maximum occurs at $\omega = \omega_o \pm \beta$.

5.43 ⋆⋆ When a car drives along a "washboard" road, the regular bumps cause the wheels to oscillate on the springs. (What actually oscillates is each axle assembly, comprising the axle and its two wheels.) Find the speed of my car at which this oscillation resonates, given the following information: **(a)** When four 80-kg men climb into my car, the body sinks by a couple of centimeters. Use this to estimate the spring constant k of each of the four springs. **(b)** If an axle assembly (axle plus two wheels) has total mass 50 kg, what is the natural frequency f of the assembly oscillating on its two springs? **(c)** If the bumps on a road are 80 cm apart, at about what speed would these oscillations go into resonance?

(a) Assuming that the weight of the four men is evenly distributed among the four springs, we can substitute $m = 80$ kg and $x = 2$ cm into the equation $mg = kx$ for any one spring. This gives $k = mg/x = 80 \times 9.8/0.02 \approx 4 \times 10^4$ N/m.

(b) Each axle assembly is attached to two springs, so the effective spring constant of its support is $2k$ and its natural frequency is

$$f = \frac{1}{2\pi}\sqrt{\frac{2k}{m}} = \frac{1}{2\pi}\sqrt{\frac{2 \times (4 \times 10^4)}{50}} \approx 6 \text{ Hz}.$$

(c) If the distance between bumps is d, then $v = fd = 6 \times 0.8 \approx 5$ m/s or roughly 10 mi/h.

5.45 ⋆⋆⋆ Consider a damped oscillator, with natural frequency ω_o and damping constant β both fixed, that is driven by a force $F(t) = F_o \cos(\omega t)$. **(a)** Find the rate $P(t)$ at which $F(t)$ does work and show that the average rate $\langle P \rangle$ over any number of complete cycles is $m\beta\omega^2 A^2$. **(b)** Verify that this is the same as the average rate at which energy is lost to the resistive force. **(c)** Show that as ω is varied $\langle P \rangle$ is maximum when $\omega = \omega_o$; that is, the resonance of the power occurs at $\omega = \omega_o$ (exactly).

The key relationships we need are

$$F(t) = F_o \cos(\omega t) \quad (5.56), \qquad x(t) = A\cos(\omega t - \delta) \quad (5.66)$$

and

$$A = (F_o/m)/\sqrt{(\omega_o^2 - \omega^2)^2 + 4\beta^2\omega^2} \quad (5.64), \qquad \tan\delta = 2\beta\omega/(\omega_o^2 - \omega^2) \quad (5.65).$$

(a) The rate at which the driving force F does work is

$$P = F\dot{x} = -F_o\omega A\cos(\omega t)\sin(\omega t - \delta) = \tfrac{1}{2}\omega A F_o[\sin\delta - \sin(2\omega t - \delta)] \tag{S5.4}$$

where, in the last equality I used the trig identity for $\cos\theta\sin\phi$. The average power over any number of complete cycles is just $\langle P \rangle = (1/\tau)\int_0^\tau P\,dt$. When we integrate the square bracket of (S5.4), the first term gives us $\tau\sin\delta$ and the second gives zero. Therefore, $\langle P \rangle = \frac{1}{2}\omega A F_o \sin\delta$. Finally, if you look at Fig.5.14, you'll see that $\sin\delta = 2\beta\omega/\sqrt{(\omega_o^2 - \omega^2)^2 + 4\beta^2\omega^2}$. So

$$\langle P \rangle = \tfrac{1}{2}\omega A F_o \sin\delta = \tfrac{1}{2}\omega A F_o \frac{2\beta\omega}{\sqrt{(\omega_o^2 - \omega^2)^2 + 4\beta^2\omega^2}} = m\beta\omega^2 A^2 \tag{S5.5}$$

where, in the last equality, I used (5.64) to replace F_o and the large square root by mA.

(b) The damping force F_{dmp} has magnitude $2m\beta v$ in the opposite direction to the velocity, so the rate of loss of energy is $P_{\text{loss}} = F_{\text{dmp}}v = 2m\beta v^2 = 2m\beta\omega^2 A^2 \sin^2(\omega t - \delta)$. When we average this over a cycle, the integration of the $\sin^2$ term gives $\tau/2$ [use the appropriate trig identity to write $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$], and we find that $\langle P_{\text{loss}} \rangle = m\beta\omega^2 A^2$, which is equal to the average power delivered, as found in part (a).

(c) Using (S5.5) and (5.64), we can write $\langle P \rangle$ in the form $\langle P \rangle = K\Omega/[(\Omega - \omega_o^2)^2 + 4\beta^2\Omega]$, where I have introduced the temporary shorthand $\Omega = \omega^2$. Now, to maximize $\langle P \rangle$ is the same as to minimize $1/\langle P \rangle = K'[4\beta^2 + (\Omega - \omega_o^2)^2/\Omega]$. The derivative of the latter is proportional to $(\Omega - \omega_o^2)$ (times a positive function). Thus the derivative vanishes when $\Omega = \omega_o^2$. As you can check, the second derivative is positive, so $1/\langle P \rangle$ has a minimum, and $\langle P \rangle$ the expected maximum, when $\omega = \omega_o$.

SECTION 5.7 Fourier Series

5.47 ⋆⋆ In order to prove the crucial formulas (5.83)–(5.85) for the Fourier coefficients a_n and b_n, you must first prove the following:

$$\int_{-\tau/2}^{\tau/2} \cos(n\omega t)\cos(m\omega t)\,dt = \begin{cases} \tau/2 & \text{if } m = n \neq 0 \\ 0 & \text{if } m \neq n. \end{cases} \tag{5.105}$$

(This integral is obviously τ if $m = n = 0$.) There is an identical result with all cosines replaced by sines, and finally

$$\int_{-\tau/2}^{\tau/2} \cos(n\omega t)\sin(m\omega t)\,dt = 0 \qquad \text{for all integers } n \text{ and } m, \tag{5.106}$$

where as usual $\omega = 2\pi/\tau$. Prove these. [*Hint:* Use trig identities to replace $\cos(\theta)\cos(\phi)$ by terms like $\cos(\theta + \phi)$ and so on.]

To treat the integrals involving two cosines, we need the trig identity $\cos\theta\cos\phi = \frac{1}{2}[\cos(\theta + \phi) + \cos(\theta - \phi)]$. For the case $n = m \neq 0$, this gives

$$\int_{-\tau/2}^{\tau/2} \cos^2(n\omega t)dt = \tfrac{1}{2}\int_{-\tau/2}^{\tau/2}[\cos(2n\omega t) + 1]dt = \tfrac{1}{2}\tau$$

because the first integral is zero [because $\sin(2n\omega t)$ is τ-periodic] and the second gives just τ. For the case $m \neq n$,, we find

$$\int_{-\tau/2}^{\tau/2} \cos(n\omega t)\cos(m\omega t)dt = \tfrac{1}{2}\int_{-\tau/2}^{\tau/2}[\cos(n+m)\omega t + \cos(n-m)\omega t]dt$$

$$= \frac{1}{2\omega}\left[\frac{\sin(n+m)\omega t}{n+m} + \frac{\sin(n-m)\omega t}{n-m}\right]_{-\tau/2}^{\tau/2} = 0$$

because both sine functions are τ periodic. Using the corresponding trig identities for $\sin\theta\sin\phi$ and $\cos\theta\sin\phi$, you can prove the corresponding results for two sine functions and for one sine and one cosine.

5.49 ⋆⋆⋆ [Computer] Find the Fourier coefficients a_n and b_n for the function shown in Figure 5.28(a). Make a plot similar to Figure 5.23, comparing the function itself with the first couple of terms in the Fourier series, and another for the first six or so terms. Take $f_{\max} = 1$.

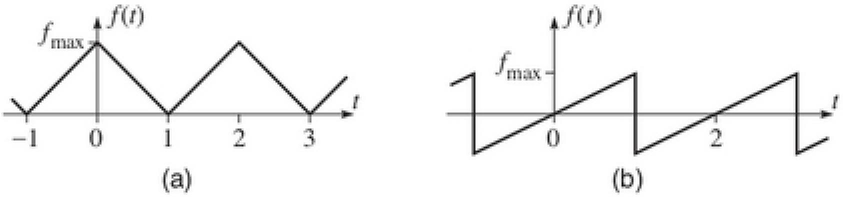

Figure 5.28 **(a)** Problem 5.49. **(b)** Problem 5.50

The given function is even, $f(-t) = +f(t)$. Therefore, $\sin(m\omega t)f(t)$ is odd and all of the integrals (5.84) for the coefficients b_m are zero. Since $\cos(m\omega t)f(t)$ is even, the coefficients a_m are not necessarily zero. Bearing in mind that $\tau = 2$, so $\omega = \pi$, we find that

$$a_0 = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)dt = \frac{f_{\max}}{2}$$

while for $m \geq 1$.

$$a_m = \frac{4}{\tau}\int_0^{\tau/2} \cos(m\omega t)f(t)dt = 2f_{\max}\int_0^1 \cos(m\pi t)(1-t)dt.$$

This integral can be evaluated (using integration by parts), and we find that

$$a_m = -\frac{2f_{\max}}{(m\pi)^2}[\cos(m\pi t)]_0^1 = \begin{cases} 0 & [m \text{ even}] \\ 4f_{\max}/(m\pi)^2 & [m \text{ odd}] \end{cases}$$

The left picture shows the sum of the first two terms (constant term plus first cosine) and the sawtooth function itself in gray. The right picture shows the first six terms; these follow the sawtooth so closely that it is hard to tell them apart except at the corners.

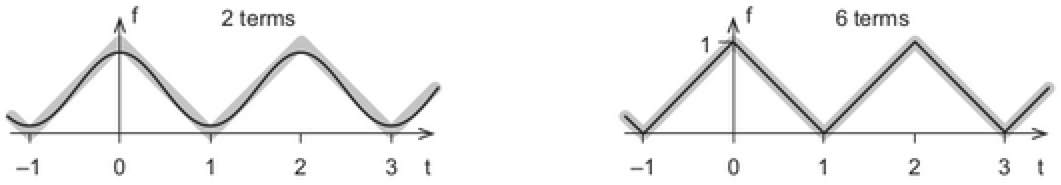

Figure S5.49

SECTION 5.8 Fourier Series Solution for the Driven Oscillator

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5.51 ⋆⋆ You can make the Fourier series solution for a periodically driven oscillator a bit tidier if you don't mind using complex numbers. Obviously the periodic force of Equation (5.90) can be written as $f = \mathrm{Re}(g)$, where the complex function g is

$$g(t) = \sum_{n=0}^{\infty} f_n e^{in\omega t}.$$

Show that the real solution for the oscillator's motion can likewise be written as $x = \mathrm{Re}(z)$, where

$$z(t) = \sum_{n=0}^{\infty} C_n e^{in\omega t}$$

and

$$C_n = \frac{f_n}{\omega_o^2 - n^2\omega^2 + 2i\beta n\omega}.$$

This solution avoids our having to worry about the real amplitude A_n and phase shift δ_n separately. (Of course A_n and δ_n are hidden inside the complex number C_n.)

If the real force is $f(t) = \sum_0^\infty f_n \cos(n\omega t)$, and we define the complex function $g(t) = \sum_0^\infty f_n e^{in\omega t}$, then it is obviously true that $f(t) = \mathrm{Re}[g(t)]$. (A major advantage of using complex functions is that by allowing the coefficients in the series for $g(t)$ to be complex, we can include the cosines *and sines* of the general Fourier series, but we'll continue to discuss a force that contains only cosine terms.) We can now try to solve the complex equation

$$Dz = \ddot{z} + 2\beta\dot{z} + \omega_o^2 z = g(t). \tag{S5.6}$$

Before we solve this, notice that, if we succeed, then it is obvious that $x(t) = \mathrm{Re}[z(t)]$ satisfies $Dx = f$. That is, the real part of $z(t)$ satisfies the actual equation of motion for the actual physical problem. To solve the complex equation (S5.6), we can try a function of the form $z(t) = \sum_0^\infty C_n e^{in\omega t}$. If we substitute this function into Eq. (S5.6), we obtain

$$Dz = \sum_0^\infty C_n(-n^2\omega^2 + 2i\beta n\omega + \omega_o^2)e^{in\omega t} = \sum_0^\infty f_n e^{in\omega t}.$$

This equation is satisfied if and only if the separate coefficients on the left are equal to the corresponding coefficients on the right, that is,

$$C_n = \frac{f_n}{\omega_o^2 - n^2\omega^2 + 2i\beta n\omega}.$$

By choosing the coefficients according to this equation, we obtain a complex solution, z, whose real part, $x = \text{Re}(z)$, is a real solution of the actual real problem. Needless to say, this is the same solution we got before, just in a slightly tidier form.

5.53 ⋆⋆⋆ [Computer] An oscillator is driven by the periodic force of Problem 5.49 [Figure 5.28(a)], which has period $\tau = 2$. **(a)** Find the long-term motion $x(t)$, assuming the following parameters: natural period $\tau_o = 2$ (that is, $\omega_o = \pi$), damping parameter $\beta = 0.1$, and maximum drive strength $f_{\max} = 1$. Find the coefficients in the Fourier series for $x(t)$ and plot the sum of the first four terms in the series for $0 \le t \le 6$. **(b)** Repeat, except with natural period equal to 3.

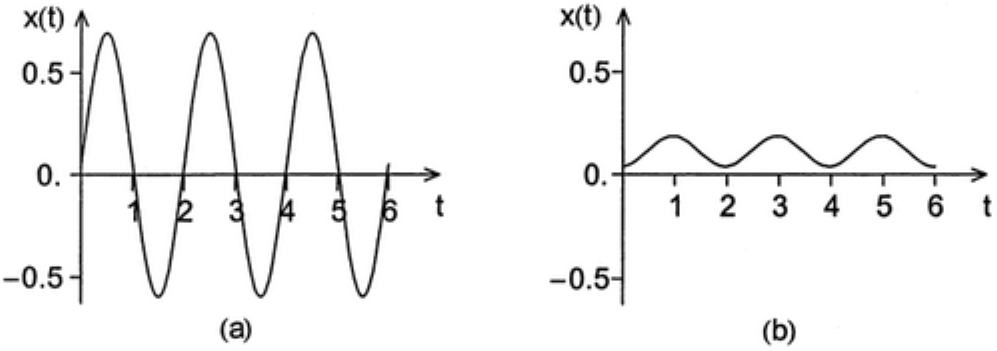
From Problem 5.49, we know that the Fourier series for $f(t)$ contains only cosine terms and that $f_0 = \frac{1}{2}$, while $f_n = 4/(n\pi)^2$ for n odd, but zero for n even. It follows that

$$x(t) = A_0 + \sum_{n\text{odd}} A_n \cos(n\omega t - \delta_n)$$

where, according to (5.92) and (5.93), $A_0 = 1/2\omega_o^2$, $\delta_0 = 0$, while for $n \ge 1$,

$$A_n = \frac{4}{n^2\pi^2\sqrt{(\omega_o^2 - n^2\omega^2)^2 + (2\beta n\omega)^2}} \qquad \text{and} \qquad \delta_n = \arctan\left(\frac{2\beta n\pi}{\omega_o^2 - n^2\pi^2}\right).$$

(a) With $\tau_o = 2$ and $\omega_o = \pi$, the first four coefficients A_n ($n = 0, 1, 2, 3$) are 0.0507, 0.6450, 0, and 0.0006. (Note the large value of A_1, because this term is on resonance.)



(a) (b)

Figure S5.53

(b) With $\tau_o = 3$ and $\omega_o = 2\pi/3$, the first four coefficients A_n ($n = 0, 1, 2, 3$) are 0.1140, 0.0734, 0, and 0.0005. (Notice that here the constant term A_0 is certainly not negligible compared to the others and it throws the oscillations off center.)

SECTION 5.9 The RMS Displacement; Parseval's Theorem

5.55 ⋆⋆ To prove the Parseval relation (5.100), one must first prove the result (5.99) for the integral of a product of cosines. Prove this result, and then use it to prove the Parseval relation.

Using the trig identity for $\cos\theta\cos\phi$ (inside front cover), we find for the integrand of Eq. (5.99)

$$\cos(n\omega t - \delta_n)\cos(m\omega t - \delta_m) = \tfrac{1}{2}\cos\big[(n+m)\omega t - (\delta_n + \delta_m)\big] + \tfrac{1}{2}\cos\big[(n-m)\omega t - (\delta_n - \delta_m)\big]$$

Let us denote by I_{nm} the integral of Eq. (5.99). If $n = m = 0$, this becomes [note that according to (5.93) $\delta_0 = 0$]

$$I_{00} = \int_{-\tau/2}^{\tau/2} 1 dt = \tau.$$

If $n = m \neq 0$, then

$$I_{nn} = \tfrac{1}{2} \int_{-\tau/2}^{\tau/2} [\cos(2n\omega t - 2\delta_n) + 1] dt = \tau/2$$

because the first integral is zero. Finally if $n \neq m$,

$$I_{nm} = \frac{1}{2\omega} \left[\frac{\sin(\cdots)}{n+m} + \frac{\sin(\cdots)}{n-m} \right]_{-\tau/2}^{\tau/2} = 0$$

because both sines are τ-periodic. This completes the proof of (5.99).

With a little rearrangement, (5.98) becomes

$$\langle x^2 \rangle = \sum_n \sum_m A_n A_m \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \cos(n\omega t - \delta_n) \cos(m\omega t - \delta_m)\, dt.$$

By the previous result, the integral here is zero unless $m = n$. Thus the double sum reduces to a single sum with $m = n$. The remaining integrals are τ (for $n = 0$) and $\tau/2$ (for $n > 0$), so we're left with $\langle x^2 \rangle = A_0^2 + \frac{1}{2} \sum_1^\infty A_n^2$, which is the Parseval relation.

5.57 ⋆⋆ [Computer] Repeat the calculations that led to Figure 5.26, using all the same parameters except taking $\beta = 0.1$. Plot your results and compare your plot with Figure 5.26.

The left picture shows the data for this problem; the right shows the data of Figure 5.26 (though drawn here to a slightly different scale). Notice that the resonances with $\beta = 0.1$ are twice as high and half as wide as those for $\beta = 0.2$.

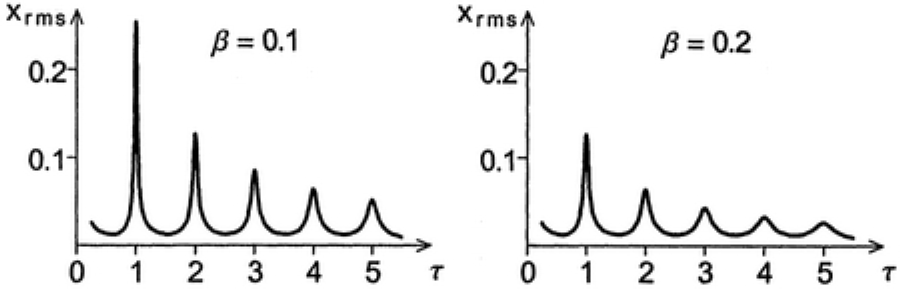


Figure S5.57

CHAPTER 6

Calculus of Variations

PROBLEMS AND SOLUTIONS

SECTION 6.1 Two Examples

6.1 ⋆ The shortest path between two points on a *curved surface*, such as the surface of a sphere, is called a **geodesic**. To find a geodesic, one has first to set up an integral that gives the length of a path on the surface in question. This will always be similar to the integral (6.2) but may be more complicated (depending on the nature of the surface) and may involve different coordinates than x and y. To illustrate this, use spherical polar coordinates (r, θ, ϕ) to show that the length of a path joining two points on a sphere of radius R is

$$L = R \int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^2\theta\, \phi'(\theta)^2}\, d\theta \tag{6.41}$$

if (θ_1, ϕ_1) and (θ_2, ϕ_2) specify the two points and we assume that the path is expressed as $\phi = \phi(\theta)$. (You will find how to minimize this length in Problem 6.16.)

Imagine first an infinitesimal section of path on the sphere, in which θ increases by $d\theta$ and ϕ by $d\phi$. This carries us a distance $R\,d\theta$ to the "south," and $R\sin\theta\, d\phi$ to the "east." The distance ds along the path is therefore

$$ds = \sqrt{(R\,d\theta)^2 + (R\sin\theta\, d\phi)^2} = R\sqrt{1 + \sin^2\theta\, \phi'(\theta)^2}\, d\theta.$$

Therefore, the total path length is $R\int_{\theta_1}^{\theta_2}\sqrt{1 + \sin^2\theta\, \phi'(\theta)^2}\, d\theta$, as claimed.

6.3 ⋆⋆ Consider a ray of light traveling in a vacuum from point P_1 to P_2 by way of the point Q on a plane mirror, as in Figure 6.8. Show that Fermat's principle implies that, on the actual path followed, Q lies in the same vertical plane as P_1 and P_2 and obeys the law of reflection, that $\theta_1 = \theta_2$. [*Hints:* Let the mirror lie in the xz plane, and let P_1 lie on the y axis at $(0, y_1, 0)$ and P_2 in the xy plane at $(x_2, y_2, 0)$. Finally let $Q = (x, 0, z)$. Calculate the time for the light to traverse the path P_1QP_2 and show that it is minimum when Q has $z = 0$ and satisfies the law of reflection.]

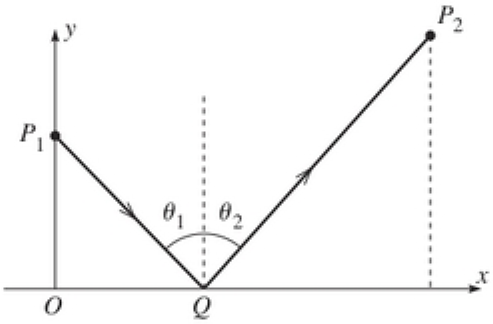


Figure 6.8 Problem 6.3

We already know that the actual path is a straight line within one medium. Therefore the segments from P_1 to Q and from Q to P_2 are straight and the corresponding distances are $P_1Q = \sqrt{x^2 + y_1^2 + z^2}$ and $QP_2 = \sqrt{(x - x_2)^2 + y_2^2 + z^2}$. Therefore the total time for the journey P_1QP_2 is

$$T = \left(\sqrt{x^2 + y_1^2 + z^2} + \sqrt{(x - x_2)^2 + y_2^2 + z^2}\right)/c.$$

To find the position of $Q = (x, 0, z)$ for which this is minimum we must differentiate with respect to z and x and set the derivatives equal to zero:

$$\frac{\partial T}{\partial z} = \frac{z}{c\sqrt{\cdots}} + \frac{z}{c\sqrt{\cdots}} = 0 \quad \Longrightarrow \quad z = 0,$$

which says that Q must lie in the same vertical plane as P_1 and P_2, and

$$\frac{\partial T}{\partial x} = \frac{x}{c\sqrt{\cdots}} + \frac{x - x_2}{c\sqrt{\cdots}} = 0 \quad \Longrightarrow \quad \sin\theta_1 = \sin\theta_2 \quad \text{or} \quad \theta_1 = \theta_2.$$

6.5★★ Fermat's principle is often stated as "the travel time of a ray of light, moving from point A to B, is minimum along the actual path." Strictly speaking it should say that the time is *stationary*, not minimum. In fact one can construct situations for which the time is maximum along the actual path. Here is one: Consider the concave, hemispherical mirror shown in Figure 6.10, with A and B at opposite ends of a diameter. Consider a ray of light traveling in a vacuum from A to B with one reflection at P, in the same vertical plane as A and B. According to the law of reflection, the actual path goes via point P_o at the bottom of the hemisphere ($\theta = 0$). Find the time of travel along the path APB as a function of θ and show that it is *maximum* at $P = P_o$. This shows the time is maximum with respect to paths of the form APB with just two straight segments. It is easy to see that it is *minimum* for other kinds of path, so the correct general statement is that it is *stationary* for arbitrary variations of the path.

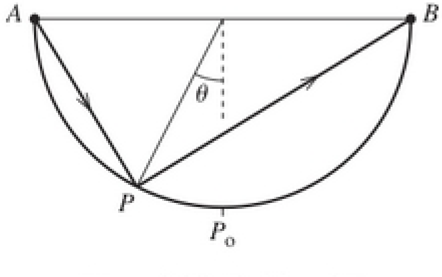


Figure 6.10 Problem 6.5

The distance from A to P is $AP = 2R\sin[(90° - \theta)/2]$, where R is the radius of the mirror. (To see this, drop a perpendicular from the center of the mirror to the line AP.) Similarly $PB = 2R\sin[(90° + \theta)/2]$. Thus the total distance APB is

$$APB = AP + PB = 2R\left(\sin\frac{90° - \theta}{2} + \sin\frac{90° + \theta}{2}\right) = 4R\sin 45° \cos\frac{\theta}{2}$$

which is *maximum* when $\theta = 0$.

SECTION 6.3 Applications of the Euler–Lagrange Equation

6.7 ⋆ Consider a right circular cylinder of radius R centered on the z axis. Find the equation giving ϕ as a function of z for the geodesic (shortest path) on the cylinder between two points with cylindrical polar coordinates (R, ϕ_1, z_1) and (R, ϕ_2, z_2). Describe the geodesic. Is it unique? By imagining the surface of the cylinder unwrapped and laid out flat, explain why the geodesic has the form it does.

The length of a small arc on the surface of the cylinder is $ds = \sqrt{R^2 d\phi^2 + dz^2} = \sqrt{R^2\phi'^2 + 1}\,dz$, where the last expression results from thinking of $\phi = \phi(z)$ as a function of z. Thus the integrand of the integral $\int ds = \int f\,dz$ is $f = \sqrt{R^2\phi'^2 + 1}$, and the Euler-Lagrange equation

$$\frac{\partial f}{\partial \phi} = \frac{d}{dz}\frac{\partial f}{\partial \phi'} \quad \text{becomes} \quad \frac{\partial f}{\partial \phi'} = \frac{R^2\phi'}{\sqrt{R^2\phi'^2 + 1}} = \text{const.}$$

If you solve this last for ϕ' you will see that $\phi' = \text{const}$ (a different constant), from which we deduce that $\phi = az + b$, with a and b chosen so that the path passes through the given points. This equation defines a path which spirals around the cylinder from (ϕ_1, z_1) to (ϕ_2, z_2), with the angle ϕ changing linearly with z.

The geodesic connecting points 1 and 2 is certainly not unique. First, we can spiral from 1 to 2 going either way (clockwise or counter-clockwise) around the cylinder, making less than one complete revolution. Further, we could spiral less steeply, making one or more complete revolutions before arriving at point 2. Generally there is a unique *shortest* path, namely the spiral on which ϕ changes by less than π, but if ϕ_1 and ϕ_2 differ by exactly π, there are two shortest paths with equal lengths. If you unwrap and flatten the cylinder, the spiral paths become straight lines, which are well known to be the shortest paths on a flat surface.

6.9 ⋆ Find the equation of the path joining the origin O to the point $P(1, 1)$ in the xy plane that makes the integral $\int_O^P (y'^2 + yy' + y^2)\,dx$ stationary.

The integrand is $f = y'^2 + yy' + y^2$, so its derivatives are $\partial f/\partial y = y' + 2y$ and $\partial f/\partial y' = 2y' + y$ and the Euler-Lagrange equation (6.13) is

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'} \quad \Longrightarrow \quad y' + 2y = 2y'' + y' \quad \Longrightarrow \quad y'' = y,$$

the general solution of which is $y(x) = A\sinh(x) + B\cosh(x)$. Since $y(0) = 0$ and $y(1) = 1$, we see that $B = 0$ and $A = 1/\sinh(1)$, so the solution is $y(x) = \sinh(x)/\sinh(1)$.

6.11 ⋆⋆ Find and describe the path $y = y(x)$ for which the integral $\int_{x_1}^{x_2} \sqrt{x}\sqrt{1+y'^2}\,dx$ is stationary.

The integrand is $f(y, y', x) = \sqrt{x}\sqrt{1+y'^2}$. Since this is independent of y, $\partial f/\partial y = 0$ and the Euler-Lagrange equation (6.13) implies simply that $\partial f/\partial y'$ is a constant; that is, $\sqrt{x}\,y'/\sqrt{1+y'^2} = k$. This can be solved for y' to give $y' = k/\sqrt{x-k^2}$, which integrates to give $y = 2k\sqrt{x-k^2} + D$, where D is a constant of integration. Rearranging we find that $x = k^2 + (y-D)^2/4k^2$, which is a parabola with it's axis along the line $y = D$.

6.13 ⋆⋆ In relativity theory, velocities can be represented by points in a certain "rapidity space" in which the distance between two neighboring points is $ds = [2/(1-r^2)]\sqrt{dr^2 + r^2\,d\phi^2}$, where r and ϕ are polar coordinates, and we consider just a two-dimensional space. (An expression like this for the distance in a non-Euclidean space is often called the *metric* of the space.) Use the Euler–Lagrange equation to show that the shortest path from the origin to any other point is a straight line.

If we write the path as $\phi = \phi(r)$, the distance from O to P is $\int_O^P ds = \int_0^R f\,dr$, where $f = [2/(1-r^2)]\sqrt{1+r^2\phi'^2}$. Since $\partial f/\partial\phi = 0$, the Euler-Lagrange equation (6.13) implies simply that $\partial f/\partial\phi'$ is a constant; that is, $[2/(1-r^2)]r^2\phi'/\sqrt{1+r^2\phi'^2} = k$. Because the path passes through the origin, $r = 0$, the constant k must in fact be zero, and we find that $\phi' = 0$. This defines a straight line through the origin.

6.15 ⋆⋆ Consider again the brachistochrone problem of Example 6.2 (page 222) but suppose that the car is launched from point 1 with initial speed v_o. Show that the path of minimum time to the fixed point 2 is still a cycloid, but with its cusp (the top point of the curve) a height $v_o^2/2g$ above point 1.

The analysis for the case that the car is launched from point 1 with fixed speed v_o is very similar to the case of Example 6.2 with $v_o = 0$, except that, by conservation of energy, the car's speed at height y is $v = \sqrt{2gy + v_o^2}$ instead of $v = \sqrt{2gy}$. If we make the change of variables from y to $\tilde{y} = y + v_o^2/2g$, the analysis goes through exactly as before and we get the same answer as in Eq. (6.26) except that it is now $\tilde{y}$ that is equal to $a(1-\cos\theta)$. Therefore $y = a(1-\cos\theta) - v_o^2/2g$ (with x as before plus a constant), and the curve is the same cycloid except that it is shifted upward and sideways.

6.17 ⋆⋆ Find the geodesics on the cone whose equation in cylindrical polar coordinates is $z = \lambda\rho$. [Let the required curve have the form $\phi = \phi(\rho)$.] Check your result for the case that $\lambda \to 0$.

The length of an element of path in cylindrical coordinates is

$$ds = \sqrt{d\rho^2 + \rho^2 d\phi^2 + dz^2} = \sqrt{(1+\lambda^2)d\rho^2 + \rho^2 d\phi^2} = \sqrt{(1+\lambda^2) + \rho^2\phi'^2}\;d\rho$$

where in the second equality I used the fact that $dz = \lambda\,d\rho$ on the cone, and in the last I assumed that the path was written as $\phi = \phi(\rho)$. Thus the path length has the standard form $L = \int f\,d\rho$, with $f = f(\phi, \phi', \rho) = \sqrt{1+\lambda^2+\rho^2\phi'^2}$. Because $\partial f/\partial\phi = 0$, the Euler-Lagrange equation reduces to

$$\frac{\partial f}{\partial\phi'} = \frac{\rho^2\phi'}{\sqrt{1+\lambda^2+\rho^2\phi'^2}} = K \quad \text{whence} \quad \phi' = \frac{K\sqrt{1+\lambda^2}}{\rho\sqrt{\rho^2-K^2}}.$$

This can be integrated (make the subtitution $K/\rho = \cos u$) to give

$$\phi - \phi_o = \sqrt{1+\lambda^2}\arccos(K/\rho) \quad \text{or} \quad \rho = \frac{K}{\cos\left[(\phi-\phi_o)/\sqrt{1+\lambda^2}\right]}.$$

This is not easily recognized as any simple curve, but in the limit that $\lambda \to 0$ the cone approaches the plane $z = 0$ and the geodesic approaches the curve $\rho = K/\cos(\phi - \phi_o)$, which you can show is a straight line perpendicular to the direction $\phi = \phi_o$, a distance K from the origin.

6.19 ⋆⋆ A surface of revolution is generated as follows: Two fixed points (x_1, y_1) and (x_2, y_2) in the x, y plane are joined by a curve $y = y(x)$. [Actually you'll make life easier if you start out writing this as $x = x(y)$.] The whole curve is now rotated about the x axis to generate a surface. Show that the curve for which the area of the surface is stationary has the form $y = y_o \cosh[(x - x_o)/y_o]$, where x_o and y_o are constants. (This is often called the soap-bubble problem, since the resulting surface is usually the shape of a soap bubble held by two coaxial rings of radii y_1 and y_2.)

The area of the surface of revolution is $A = \int 2\pi y\, ds = 2\pi \int_{y_1}^{y_2} y\sqrt{1 + x'^2}dy$, which we can write as $A = 2\pi \int f\, dy$ where $f = y\sqrt{1 + x'^2}$. Because $\partial f/\partial x = 0$, the Euler-Lagrange equation reads

$$\frac{\partial f}{\partial x'} = \frac{yx'}{\sqrt{1 + x'^2}} = y_o \quad \text{whence} \quad x' = \frac{y_o}{\sqrt{y^2 - y_o^2}},$$

where y_o is a constant. Using the substitution $y/y_o = \cosh u$, you can integrate this to give $x - x_o = y_o \operatorname{arccosh}(y/y_o)$ (where x_o is another constant) or $y = y_o \cosh[(x - x_o)/y_o]$.

6.21 ⋆⋆ In Example 6.2 (page 222) we found the brachistochrone by exchanging the variables x and y. Here is a method that avoids that exchange: Write the time as in Equation (6.19) but using x as the variable of integration. Your integrand should have the form $f(y, y', x) = \sqrt{(y'^2 + 1)/y}$. Since this is independent of x, you can invoke the "first integral" (6.43) of Problem 6.20. Show that this differential equation leads you to the same integral for x as in Equation (6.23) and hence to the same curve as before.

The time for the car to go from 1 to 2 is $t = \int_1^2 ds/v$. As usual, $ds = \sqrt{1 + y'^2}\, dx$, and, as before, $v = \sqrt{2gy}$. Therefore, $t = (1/\sqrt{2g}) \int f\, dx$, where $f = \sqrt{(1 + y'^2)/y}$. Because f does not depend explicitly on x (that is, $\partial f/\partial x = 0$), we can use the first integral (6.43) from Problem 6.20 and obtain

$$f - y'\frac{\partial f}{\partial y'} = \frac{\sqrt{1 + y'^2}}{\sqrt{y}} - y'\frac{y'}{\sqrt{y}\sqrt{1 + y'^2}} = \frac{1}{\sqrt{y}\sqrt{1 + y'^2}} = k.$$

Solving for y' we find

$$\frac{dy}{dx} = \sqrt{\frac{1}{k^2 y} - 1} = \sqrt{\frac{2a - y}{y}}$$

if we rename $k^2 = 1/(2a)$. Separating the variables x and y and integrating gives $x = \int \sqrt{y/(2a - y)}\, dy$, which is precisely the result (6.23). From here on, the derivation of the brachistochrone is exactly the same as in Example 6.2.

6.23 ⋆⋆⋆ An aircraft whose airspeed is v_o has to fly from town O (at the origin) to town P, which is a distance D due east. There is a steady gentle wind shear, such that $\mathbf{v}_{\text{wind}} = V y\, \hat{\mathbf{x}}$, where x and y are measured east and north respectively. Find the path, $y = y(x)$, which the plane should follow to minimize its flight time, as follows: **(a)** Find the plane's ground speed in terms of v_o, V, ϕ (the angle by which the plane heads to the north of east), and the plane's position. **(b)** Write down the time of flight as an integral of the form $\int_0^D f\, dx$. Show that if we assume that y' and ϕ both remain small (as is certainly reasonable if the wind speed is not too large), then the integrand f takes the approximate form $f = (1 + \frac{1}{2}y'^2)/(1 + ky)$ (times an uninteresting constant) where $k = V/v_o$. **(c)** Write down the Euler–Lagrange equation that determines the best path. To solve it, make the intelligent guess that $y(x) = \lambda x(D - x)$, which clearly passes through the two towns. Show that it satisfies the Euler–Lagrange equation, provided $\lambda = (\sqrt{4 + 2k^2D^2} - 2)/(kD^2)$. How far north does this path take the plane, if $D = 2000$ miles, $v_o = 500$ mph, and the wind shear is $V = 0.5$ mph/mi? How much time does the plane save by following this path? [You'll probably want to use a computer to do this integral.]

(a) If the plane is at position (x, y) and is heading at an angle ϕ north of east, its velocity relative to the air is $(v_o \cos\phi, v_o \sin\phi)$, while the wind's velocity is $(Vy, 0)$. The plane's velocity relative to the ground is the sum of these two vectors, so its ground speed is

$$v = \sqrt{(v_o \cos\phi + Vy)^2 + (v_o \sin\phi)^2} = \sqrt{v_o^2 + 2v_oVy\cos\phi + V^2y^2} \approx v_o + Vy = v_o(1 + ky)$$

where I have made the approximations that $Vy \ll v_o$ and $\cos\phi \approx 1$ (assuming that ϕ is always small) and where $k = V/v_o$.

(b) The time of flight is

$$t = \int_O^P \frac{ds}{v} = \int_0^D \frac{\sqrt{1 + y'^2}\,dx}{v} \approx \int_0^D \frac{1 + \frac{1}{2}y'^2}{v_o(1 + ky)}dx = \frac{1}{v_o}\int_0^D f\,dx \tag{S6.1}$$

where I used the binomial approximation in the numerator (assuming y' is always small) and where $f = (1 + \frac{1}{2}y'^2)/(1 + ky)$.

(c) With a little algebra, you can show that the Euler-Lagrange equation is

$$y''(1 + ky) - \tfrac{1}{2}ky'^2 + k = 0.$$

If you substitute the proposed guess, $y = \lambda x(D - x)$, this becomes (after a little more algebra) $kD^2\lambda^2 + 4\lambda - 2k = 0$, of which the relevant solution is $\lambda = (\sqrt{4 + 2k^2D^2} - 2)/(kD^2)$. (The second solution is negative which makes no sense.)

The maximum displacement to the north is $y_{\max} = \lambda D^2/4$, which, if we put in the given numbers, is $y_{\max} = 366$ miles. If we put the same numbers in Eq. (S6.1) and do the integral numerically, we find for the time of flight $t = 3.556$ hours. The time along the direct path would be 4 hours, so the saving is 0.444 hours or 27 minutes.

6.25 ⋆⋆⋆ Consider a single loop of the cycloid (6.26) with a fixed value of a, as shown in Figure 6.11. A car is released from rest at a point P_o anywhere on the track between O and the lowest point P (that is, P_o has parameter $0 < \theta_o < \pi$). Show that the time for the cart to roll from P_o to P is given by the integral

$$\text{time}(P_o \to P) = \sqrt{\frac{a}{g}}\int_{\theta_o}^{\pi}\sqrt{\frac{1 - \cos\theta}{\cos\theta_o - \cos\theta}}\,d\theta$$

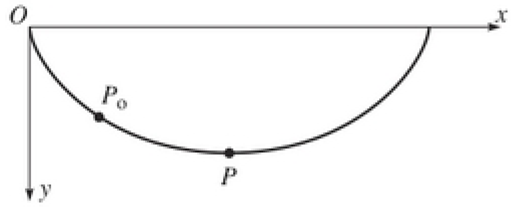


Figure 6.11 Problem 6.25

and prove that this time is equal to $\pi\sqrt{a/g}$. Since this is *independent of the position of* P_o, the cart takes the same time to roll from P_o to P, whether P_o is at O, or anywhere between O and P, even infinitesimally close to P. Explain qualitatively how this surprising result can possibly be true. [*Hint:* To do the mathematics, you have to make some cunning changes of variables. One route is this: Write $\theta = \pi - 2\alpha$ and then use the relevant trig identities to replace the cosines of θ by sines of α. Now substitute $\sin\alpha = u$ and do the remaining integral.]

The parametric equation (6.26) of the cycloid is $x = a(\theta - \sin\theta)$, $\quad y = a(1 - \cos\theta)$, from which we find the derivatives $x' = a(1 - \cos\theta)$ and $y' = a\sin\theta$. Therefore, the element of path length is

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{x'^2 + y'^2}\,d\theta = a\sqrt{(1-\cos\theta)^2 + \sin^2\theta}\,d\theta = a\sqrt{2(1-\cos\theta)}\,d\theta.$$

The speed of the cart is given by conservation of energy as

$$v = \sqrt{2g(y - y_\text{o})} = \sqrt{2ga(\cos\theta_\text{o} - \cos\theta)}.$$

Therefore the required time is

$$t = \int_{P_\text{o}}^{P} \frac{ds}{v} = \int_{\theta_\text{o}}^{\pi} \frac{a\sqrt{2(1-\cos\theta)}}{\sqrt{2ga(\cos\theta_\text{o} - \cos\theta)}}\,d\theta.$$

If we make the substitution $\theta = \pi - 2\alpha$ and use a couple of trig identities, this becomes

$$t = 2\sqrt{\frac{a}{g}}\int_0^{\alpha_\text{o}} \frac{\cos\alpha}{\sqrt{\sin^2\alpha_\text{o} - \sin^2\alpha}}\,d\alpha.$$

Finally, the substitutions $\sin\alpha = u$ and then $u/u_\text{o} = v$ reduce this to

$$t = 2\sqrt{\frac{a}{g}}\int_0^{u_\text{o}} \frac{du}{\sqrt{u_\text{o}^2 - u^2}} = 2\sqrt{\frac{a}{g}}\int_0^1 \frac{dv}{\sqrt{1 - v^2}} = \pi\sqrt{\frac{a}{g}}$$

which *is* independent of θ_o.

The higher the starting point P_o, the further the car has to go, but the steeper the initial slope and the faster the car goes. On a cycloid, these two effects perfectly cancel, so that the time to reach P is independent of the position of P_o.

SECTION 6.4 More than Two Variables

6.27 ★★ Prove that the shortest path between two points in three dimensions is a straight line. Write the path in the parametric form

$$x = x(u), \qquad y = y(u), \qquad \text{and} \qquad z = z(u)$$

and then use the three Euler–Lagrange equations corresponding to (6.34).

The element of path length is $ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{x'^2 + y'^2 + z'^2}\,du$. Thus the total path length is $L = \int f\,du$ where $f = \sqrt{x'^2 + y'^2 + z'^2}$. There are three Euler-Lagrange equations, which involve the following six derivatives:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0,$$

and

$$\frac{\partial f}{\partial x'} = \frac{x'}{\sqrt{x'^2 + y'^2 + z'^2}}, \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{x'^2 + y'^2 + z'^2}}, \quad \frac{\partial f}{\partial z'} = \frac{z'}{\sqrt{x'^2 + y'^2 + z'^2}}.$$

Since the first three derivatives are zero, the Euler-Lagrange equations imply simply that each of the last three is constant. This means that the ratios, $x' : y' : z'$ are constant, which implies in turn that as we move along the curve the ratios $dx : dy : dz$ are constant. In other words, the curve is a straight line.

CHAPTER 7

Lagrange's Equations

PROBLEMS AND SOLUTIONS

SECTION 7.1 Lagrange's Equations for Unconstrained Motion

7.1 ⋆ Write down the Lagrangian for a projectile (subject to no air resistance) in terms of its Cartesian coordinates (x, y, z), with z measured vertically upward. Find the three Lagrange equations and show that they are exactly what you would expect for the equations of motion.

The kinetic and potential energies are

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \quad \text{and} \quad U = mgz.$$

Therefore

$$\mathcal{L} = T - U = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

The three Lagrange equations (7.7) are

$$0 = m\ddot{x}, \quad 0 = m\ddot{y}, \quad \text{and} \quad -mg = m\ddot{z},$$

which are just the three components of the equation $\mathbf{F} = m\mathbf{a}$ for a projectile with $\mathbf{F} = m\mathbf{g}$.

7.3 ⋆ Consider a mass m moving in two dimensions with potential energy $U(x, y) = \frac{1}{2}kr^2$, where $r^2 = x^2 + y^2$. Write down the Lagrangian, using coordinates x and y, and find the two Lagrange equations of motion. Describe their solutions. [This is the potential energy of an ion in an "ion trap," which can be used to study the properties of individual atomic ions.]

$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}k(x^2 + y^2)$ and the two Lagrange equations are $-kx = m\ddot{x}$ and $-ky = m\ddot{y}$. In the general solution, x and y oscillate with the same angular frequency $\omega = \sqrt{k/m}$ and the point (x, y) moves around an ellipse.

7.5 ⋆ Find the components of $\nabla f(r, \phi)$ in two-dimensional polar coordinates. [*Hint:* Remember that the change in the scalar f as a result of an infinitesimal displacement $d\mathbf{r}$ is $df = \nabla f \cdot d\mathbf{r}$.]

If we make a small displacement from $\mathbf{r}$ to $\mathbf{r} + d\mathbf{r}$, we know from (4.35) that the change in $f(\mathbf{r})$ is $df = \nabla f \cdot d\mathbf{r}$. In two-dimensional polars, $d\mathbf{r} = (dr, r\,d\phi)$. Therefore,

$$df = (\nabla f)_r\,dr + (\nabla f)_\phi\,r\,d\phi.$$

On the other hand, we know from two-variable calculus that

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \phi}d\phi.$$

Both of these are valid for arbitrary (small) values of dr and $d\phi$, so we can compare coefficients to give

$$(\nabla f)_r = \frac{\partial f}{\partial r} \qquad \text{and} \qquad (\nabla f)_\phi = \frac{1}{r}\frac{\partial f}{\partial \phi}.$$

7.6 ⋆ Consider two particles moving unconstrained in three dimensions, with potential energy $U(\mathbf{r}_1, \mathbf{r}_2)$. **(a)** Write down the six equations of motion obtained by applying Newton's second law to each particle. **(b)** Write down the Lagrangian $\mathcal{L}(\mathbf{r}_1, \mathbf{r}_2, \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2) = T - U$ and show that the six Lagrange equations are the same as the six Newtonian equations of part (a). This establishes the validity of Lagrange's equations in rectangular coordinates, which in turn establishes Hamilton's principle. Since the latter is independent of coordinates, this proves Lagrange's equations in any coordinate system.

7.7 ⋆ Do Problem 7.6, but for N particles moving unconstrained in three dimensions (in which case there are $3N$ equations of motion).

(a) Newton's second law gives the N vector equations (or $3N$ scalar equations)

$$\mathbf{F}_\alpha = -\nabla_\alpha U = m_\alpha \ddot{\mathbf{r}}_\alpha \quad [\alpha = 1, \cdots, N]. \tag{S7.1}$$

(b) The Lagrangian is

$$\mathcal{L}(\mathbf{r}_1, \cdots, \mathbf{r}_N, \dot{\mathbf{r}}_1, \cdots, \dot{\mathbf{r}}_N) = T - U = \sum_1^N \tfrac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha^2 - U(\mathbf{r}_1, \cdots, \mathbf{r}_N)$$

and the $3N$ Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_1} \quad \Longrightarrow \quad -\partial U/\partial x_1 = m_1 \ddot{x}_1$$

and so on. These are just the $3N$ components of the N Newtonian equations (S7.1).

SECTION 7.3 Constrained Systems in General

7.9 ⋆ Consider a bead that is threaded on a rigid circular hoop of radius R lying in the xy plane with its center at O, and use the angle ϕ of two-dimensional polar coordinates as the one generalized coordinate to describe the bead's position. Write down the equations that give the Cartesian coordinates (x, y) in terms of ϕ and the equation that gives the generalized coordinate ϕ in terms of (x, y).

The requested equations are just the standard equations for two-dimensional polar coordinates, with the radius fixed at that of the hoop, $r = R$, namely $x = R\cos\phi$, $\quad y = R\sin\phi$ and, in the other direction, $\phi = \arctan(y/x)$ with ϕ chosen to lie in the right quadrant.

7.11 ⋆ Consider the pendulum of Figure 7.4, suspended inside a railroad car, but suppose that the car is oscillating back and forth, so that the point of suspension has position $x_s = A\cos\omega t,\ y_s = 0$. Use the angle ϕ as the generalized coordinate and write down the equations that give the Cartesian coordinates of the bob in terms of ϕ and vice versa.

$x = x_s + l\sin\phi = A\cos(\omega t) + l\sin\phi$, and $y = y_s + l\cos\phi = l\cos\phi$. In the other direction, $\phi = \arctan[(x - A\cos\omega t)/y]$.

SECTION 7.4 Proof of Lagrange's Equations with Constraints

7.13 ⋆⋆ In Section 7.4 [Equations (7.41) through (7.51)], I proved Lagrange's equations for a single particle constrained to move on a two-dimensional surface. Go through the same steps to prove Lagrange's equations for a system consisting of two particles subject to various unspecified constraints. [*Hint:* The net force on particle 1 is the sum of the total constraint force $\mathbf{F}_1^{\rm cstr}$ and the total nonconstraint force $\mathbf{F}_1$, and likewise for particle 2. The constraint forces come in many guises (the normal force of a surface, the tension force of a string tied between the particles, etc.), but it is always true that the net work done by all constraint forces in any displacement consistent with the constraints is zero — this is the defining property of constraint forces. Meanwhile, we take for granted that the nonconstraint forces are derivable from a potential energy $U(\mathbf{r}_1, \mathbf{r}_2, t)$; that is, $\mathbf{F}_1 = -\nabla_1 U$ and likewise for particle 2. Write down the difference δS between the action integral for the right path given by $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ and any nearby wrong path given by $\mathbf{r}_1(t) + \boldsymbol{\epsilon}_1(t)$ and $\mathbf{r}_2(t) + \boldsymbol{\epsilon}_2(t)$. Paralleling the steps of Section 7.4, you can show that δS is given by an integral analogous to (7.49), and this is zero by the defining property of constraint forces.]

The Lagrangian is $\mathcal{L} = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 - U(\mathbf{r}_1, \mathbf{r}_2, t)$. Suppose that in the actual motion the two particles follow the "right" path $\mathbf{r}_1 = \mathbf{r}_1(t)$ and $\mathbf{r}_2 = \mathbf{r}_2(t)$, and we'll compare the action along this path with that on the "wrong" one $\mathbf{R}_1(t) = \mathbf{r}_1(t) + \boldsymbol{\epsilon}_1(t)$ and $\mathbf{R}_2(t) = \mathbf{r}_2(t) + \boldsymbol{\epsilon}_2(t)$. The difference between the Lagrangians on these two paths is

$$\begin{aligned}
\delta\mathcal{L} &= \mathcal{L}(\mathbf{R}_1, \mathbf{R}_2, \dot{\mathbf{R}}_1, \dot{\mathbf{R}}_2, t) - \mathcal{L}(\mathbf{r}_1, \mathbf{r}_2, \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2, t)\\
&= \tfrac{1}{2}m_1[(\dot{\mathbf{r}}_1 + \dot{\boldsymbol{\epsilon}}_1)^2 - \dot{\mathbf{r}}_1^2] + \tfrac{1}{2}m_2[(\dot{\mathbf{r}}_2 + \dot{\boldsymbol{\epsilon}}_2)^2 - \dot{\mathbf{r}}_2^2] - [U(\mathbf{r}_1 + \boldsymbol{\epsilon}_1, \mathbf{r}_2 + \boldsymbol{\epsilon}_2, t) - U(\mathbf{r}_1, \mathbf{r}_2, t)]\\
&\approx m_1\dot{\mathbf{r}}_1 \cdot \dot{\boldsymbol{\epsilon}}_1 + m_2\dot{\mathbf{r}}_2 \cdot \dot{\boldsymbol{\epsilon}}_2 - \boldsymbol{\epsilon}_1 \cdot \nabla_1 U - \boldsymbol{\epsilon}_2 \cdot \nabla_2 U.
\end{aligned}$$

Therefore, the difference between the two action integrals is

$$\begin{aligned}
\delta S = \int \delta\mathcal{L}\,dt &= -\int [\boldsymbol{\epsilon}_1 \cdot (m_1\ddot{\mathbf{r}}_1 + \nabla_1 U) + \boldsymbol{\epsilon}_2 \cdot (m_2\ddot{\mathbf{r}}_2 + \nabla_2 U)]dt\\
&= -\int [\boldsymbol{\epsilon}_1 \cdot \mathbf{F}_1^{\rm cstr} + \boldsymbol{\epsilon}_2 \cdot \mathbf{F}_2^{\rm cstr}]dt.
\end{aligned}$$

(In the first line I used integration by parts to move a time derivative from each $\dot{\boldsymbol{\epsilon}}$ to the corresponding $\dot{\mathbf{r}}$, and in the second I used Newton's second law.) The integral on the second line is the work done by the constraint forces in the displacement from $\mathbf{r}_1$ to $\mathbf{r}_1 + \boldsymbol{\epsilon}_1$ and from $\mathbf{r}_2$ to $\mathbf{r}_2 + \boldsymbol{\epsilon}_2$. Provided this displacement is consistent with the constraints, this work is zero. Thus we have proved that the action integral is stationary for any displacement of path consistent with the constraints. If we now introduce generalized coordinates $q_1, \cdots, q_n$, then *any* variation of $q_1, \cdots, q_n$ is consistent with the constraints. Therefore the action integral $S = \int \mathcal{L}(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n, t)dt$ is stationary for any variations of $q_1, \cdots, q_n$, and the correct path must satisfy the n Euler-Lagrange equations (7.52).

SECTION 7.5 Examples of Lagrange's Equations

7.15 ⋆ A mass m_1 rests on a frictionless horizontal table and is attached to a massless string. The string runs horizontally to the edge of the table, where it passes over a massless, frictionless pulley and then hangs vertically down. A second mass m_2 is now attached to the bottom end of the string. Write down the Lagrangian for the system. Find the Lagrange equation of motion, and solve it for the acceleration of the blocks. For your generalized coordinate, use the distance x of the second mass below the tabletop.

Since both masses have the same speed $\dot{x}$, the total KE is $T = \frac{1}{2}(m_1 + m_2)\dot{x}^2$, whereas the PE is due to the second mass alone, $U = -m_2 g x$. Therefore,

$$\mathcal{L} = T - U = \tfrac{1}{2}(m_1 + m_2)\dot{x}^2 + m_2 g x.$$

The Lagrange equation is $m_2 g = (m_1 + m_2)\ddot{x}$, from which it follows $\ddot{x} = g m_2/(m_1 + m_2)$.

7.17 ⋆ Use the Lagrangian method to find the acceleration of the Atwood machine of Example 7.3 (page 255) including the effect of the pulley's having moment of inertia I. (The kinetic energy of the pulley is $\frac{1}{2}I\omega^2$, where ω is its angular velocity.)

The KE of rotation of the pulley is $\frac{1}{2}I\omega^2 = \frac{1}{2}I\dot{x}^2/R^2$ since $\omega = \dot{x}/R$. Therefore, the total KE is $T = \frac{1}{2}(m_1 + m_2 + I/R^2)\dot{x}^2$, while the PE is $U = -(m_1 - m_2)gx$ as before. Thus the Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}(m_1 + m_2 + I/R^2)\dot{x}^2 + (m_1 - m_2)gx$$

and the Lagrange equation is

$$(m_1 - m_2)g = (m_1 + m_2 + I/R^2)\ddot{x}.$$

That is, $\ddot{x} = (m_1 - m_2)g/(m_1 + m_2 + I/R^2)$.

7.19 ⋆ In Example 7.5 (page 258) the two accelerations are given by Equations (7.66) and (7.67). Check that the acceleration of the block is given correctly in the limit $M \to 0$. [You need to find the components of this acceleration *relative to the table*.]

With its mass $M = 0$, the presence of the wedge should be irrelevant; that is, the block should fall straight down with acceleration g, pushing the massless wedge aside as it goes. To verify this, note that the position of the block relative to the table is $\mathbf{r} = (q_2 + q_1\cos\alpha, q_1\sin\alpha)$ (with axes chosen as in Figure 7.8). Therefore the acceleration is

$$\mathbf{a} = (\ddot{q}_2 + \ddot{q}_1\cos\alpha, \ddot{q}_1\sin\alpha) = \ddot{q}_1\left(\left[1 - \frac{m}{M+m}\right]\cos\alpha, \sin\alpha\right) \tag{S7.2}$$

where in the last expression I have replaced $\ddot{q}_2$ using Equation (7.66). As $M \to 0$, the x component approaches zero, as expected. To see what happens to the y component, we use (7.67) for $\ddot{q}_1$, which, in the limit that $M = 0$, gives

$$\ddot{q}_1 = \frac{g\sin\alpha}{1 - \cos^2\alpha} = \frac{g}{\sin\alpha}.$$

Inserting this into Eq. (S7.2), we find that, when $M = 0$, the y component of $\mathbf{a}$ is just g, as anticipated.

7.21 ⋆ The center of a long frictionless rod is pivoted at the origin, and the rod is forced to rotate in a horizontal plane with constant angular velocity ω. Write down the Lagrangian for a bead of mass m threaded on the rod, using r as your generalized coordinate, where r, ϕ are the polar coordinates of the bead. (Notice that ϕ is not an independent variable since it is fixed by the rotation of the rod to be $\phi = \omega t$.) Solve Lagrange's equation for $r(t)$. What happens if the bead is initially at rest at the origin? If it is released from any point $r_o > 0$, show that $r(t)$ eventually grows exponentially. Explain your results in terms of the centrifugal force $m\omega^2 r$.

If we use two-dimensional polar coordinates, the bead's velocity is $\mathbf{v} = (\dot{r}, r\dot{\phi}) = (\dot{r}, r\omega)$, where ω is the fixed angular velocity with which the rod is forced to rotate. Thus $\mathcal{L} = T - U = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\omega^2)$. ($U$ is a constant, which we may as well take to be zero.) The Lagrange equation is $\ddot{r} = \omega^2 r$, the general solution of which is $r(t) = Ae^{\omega t} + Be^{-\omega t}$. If $r(0) = \dot{r}(0) = 0$, then $A = B = 0$ and the bead stays put; that is, $r = 0$ is an equilibrium point (though unstable, as we'll see). If $r(0) = r_o \neq 0$, but $\dot{r}(0) = 0$, then $A = B = r_o/2$ and

$$r(t) = \tfrac{1}{2}r_o(e^{\omega t} + e^{-\omega t}) \to \tfrac{1}{2}r_o e^{\omega t}$$

as $t \to \infty$. As seen in the rotating frame of the rod, there is an ouward "centrifugal force" $m\omega^2 r$. This causes the bead to accelerate outward, and as r increases, the acceleration increases in proportion—hence the exponential growth of r.

7.23 ⋆ A small cart (mass m) is mounted on rails inside a large cart. The two are attached by a spring (force constant k) in such a way that the small cart is in equilibrium at the midpoint of the large. The distance of the small cart from its equilibrium is denoted x and that of the large one from a fixed point on the ground is X, as shown in Figure 7.13. The large cart is now forced to oscillate such that $X = A\cos\omega t$, with both A and ω fixed. Set up the Lagrangian for the motion of the small cart and show that the Lagrange equation has the form

$$\ddot{x} + \omega_o^2 x = B\cos\omega t$$

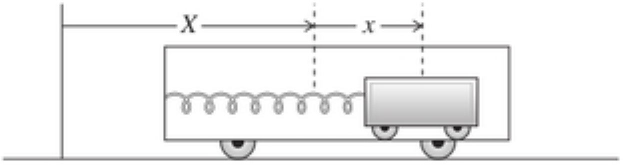


Figure 7.13 Problem 7.23

where ω_o is the natural frequency $\omega_o = \sqrt{k/m}$ and B is a constant. This is the form assumed in Section 5.5, Equation (5.57), for driven oscillations (except that we are here ignoring damping). Thus the system described here would be one way to realize the motion discussed there. (We could fill the large cart with molasses to provide some damping.)

The small cart's KE is $T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x} + \dot{X})^2 = \frac{1}{2}m(\dot{x} - A\omega\sin\omega t)^2$, and $U = \frac{1}{2}kx^2$. Thus $\partial\mathcal{L}/\partial\dot{x} = m(\dot{x} - A\omega\sin\omega t)$ and Lagrange's equation reads

$$-kx = m\ddot{x} - mA\omega^2\cos\omega t \quad \text{or} \quad \ddot{x} + \omega_o^2 x = B\cos\omega t$$

where I have replaced k/m by ω_o^2 and renamed $A\omega^2$ as B.

7.25 ⋆ Prove that the potential energy of a central force $\mathbf{F} = -kr^n\hat{\mathbf{r}}$ (with $n \neq -1$) is $U = kr^{n+1}/(n+1)$ if we choose the zero of U appropriately. In particular, if $n = 1$, then $\mathbf{F} = -k\mathbf{r}$ and $U = \frac{1}{2}kr^2$.

If $\mathbf{F} = -kr^n\hat{\mathbf{r}}$, then $U(\mathbf{r}) = -\int_{\mathbf{r}_o}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}' = \int_{r_o}^{r} kr'^n dr' = kr^{n+1}/(n+1)$ (plus a constant that we can choose to be zero).

7.27 ⋆⋆ Consider a double Atwood machine constructed as follows: A mass $4m$ is suspended from a string that passes over a massless pulley on frictionless bearings. The other end of this string supports a second similar pulley, over which passes a second string supporting a mass of $3m$ at one end and m at the other. Using two suitable generalized coordinates, set up the Lagrangian and use the Lagrange equations to find the acceleration of the mass $4m$ when the system is released. Explain why the top pulley rotates even though it carries equal weights on each side.

Let x be the distance from the top pulley down to the lower one and y that from the lower pulley to the mass $3m$. Then

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}m\left[4\dot{x}^2 + (\dot{x}-\dot{y})^2 + 3(\dot{x}+\dot{y})^2\right] - mg[4x + (y-x) - 3(x+y)]\\ &= 2m(2\dot{x}^2 + \dot{x}\dot{y} + \dot{y}^2) + 2mgy.\end{aligned}$$

The x equation is $0 = 4\ddot{x} + \ddot{y}$ and the y equation is $g = \ddot{x} + 2\ddot{y}$, from which we find $\ddot{x} = -g/7$. That is, the mass $4m$ accelerates down at $g/7$.

If the top pulley were stationary, then the same would be true of the mass $4m$. Thus the tension in the top string would be $4mg$. This would mean that the net force on the system consisting of the lower pulley and its two masses would be zero, and hence that its CM could not be accelerating. But with the upper string stationary, the lower string would behave just like a single Atwood machine and the CM of the two masses m and $3m$ would clearly accelerate down. This is a contradiction, so the top pulley has to move (and, in fact, accelerate).

7.29 ⋆⋆ Figure 7.14 shows a simple pendulum (mass m, length l) whose point of support P is attached to the edge of a wheel (center O, radius R) that is forced to rotate at a fixed angular velocity ω. At $t = 0$, the point P is level with O on the right. Write down the Lagrangian and find the equation of motion for the angle ϕ. [*Hint:* Be careful writing down the kinetic energy T. A safe way to get the velocity right is to write down the position of the bob at time t, and then differentiate.] Check that your answer makes sense in the special case that $\omega = 0$.

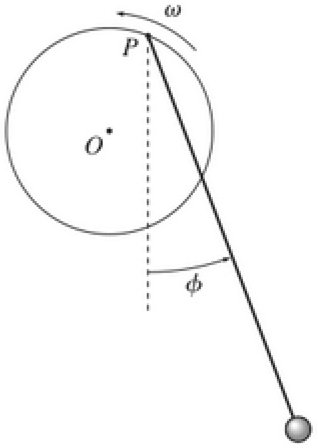


Figure 7.14 Problem 7.29

Because the angle between the line OP and the horizontal is ωt, the position of P is ($R\cos\omega t$, $R\sin\omega t$). Therefore the position of the pendulum's bob is

$$\mathbf{r} = (x, y) = (R\cos\omega t + l\sin\phi,\ R\sin\omega t - l\cos\phi)$$

and its velocity is

$$\mathbf{v} = (\dot{x}, \dot{y}) = (-\omega R\sin\omega t + \dot{\phi} l\cos\phi, \omega R\cos\omega t + \dot{\phi} l\sin\phi).$$

Therefore the Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}mv^2 - mgy = \tfrac{1}{2}m[\omega^2 R^2 + \dot{\phi}^2 l^2 + 2\omega R\dot{\phi} l\sin(\phi - \omega t)] - mg(R\sin\omega t - l\cos\phi)$$

where I have used a couple of trig identities to combine various terms (and omitted a constant). The two derivatives of $\mathcal{L}$ are

$$\frac{\partial\mathcal{L}}{\partial\phi} = m\omega R\dot{\phi} l\cos(\phi - \omega t) - mgl\sin\phi \qquad \text{and} \qquad \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = m[\dot{\phi} l^2 + \omega Rl\sin(\phi - \omega t)]$$

and the Lagrange equation, after a couple of cancellations, is

$$l\ddot{\phi} = -g\sin\phi + \omega^2 R\cos(\phi - \omega t).$$

As $\omega \to 0$, this becomes $l\ddot{\phi} = -g\sin\phi$, the equation for an ordinary simple pendulum.

7.31 ⋆⋆ A simple pendulum (mass M and length L) is suspended from a cart (mass m) that can oscillate on the end of a spring of force constant k, as shown in Figure 7.15. **(a)** Write the Lagrangian in terms of the two generalized coordinates x and ϕ, where x is the extension of the spring from its equilibrium length. (Read the hint in Problem 7.29.) Find the two Lagrange equations. (Warning: They're pretty ugly!) **(b)** Simplify the equations to the case that both x and ϕ are small. (They're still pretty ugly, and note, in particular, that they are still *coupled*; that is, each equation involves both variables. Nonetheless, we shall see how to solve these equations in Chapter 11 — see particularly Problem 11.19.)

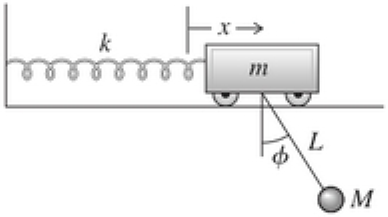


Figure 7.15 Problem 7.31

(a) The position and hence velocity of the mass M are

$$\mathbf{r}_M = (x + L\sin\phi, L\cos\phi) \qquad \text{and} \qquad \mathbf{v}_M = (\dot{x}_M, \dot{y}_M) = (\dot{x} + L\dot{\phi}\cos\phi, -L\dot{\phi}\sin\phi).$$

Therefore the Lagrangian is

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}(mv_m^2 + Mv_M^2) + Mgy_M - \tfrac{1}{2}kx^2 \\ &= \tfrac{1}{2}(m + M)\dot{x}^2 + \tfrac{1}{2}M(L^2\dot{\phi}^2 + 2\dot{x}L\dot{\phi}\cos\phi) + MgL\cos\phi - \tfrac{1}{2}kx^2\end{aligned}$$

and the two Lagrange equations are (after a little tidying up)

$$\frac{\partial\mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} \Rightarrow (m + M)\ddot{x} + ML\ddot{\phi}\cos\phi - ML\dot{\phi}^2\sin\phi = -kx$$

and

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \Rightarrow L\ddot{\phi} + \ddot{x}\cos\phi = -g\sin\phi$$

(b) If ϕ remains small, we can write $\cos\phi \approx 1$ and $\sin\phi \approx \phi$ and ignore powers of ϕ or $\dot{\phi}$ higher than second to give

$$(m+M)\ddot{x} + ML\ddot{\phi} = -kx \qquad \text{and} \qquad L\ddot{\phi} + \ddot{x} = -g\phi.$$

7.33 ⋆⋆ A bar of soap (mass m) is at rest on a frictionless rectangular plate that rests on a horizontal table. At time $t = 0$, I start raising one edge of the plate so that the plate pivots about the opposite edge with constant angular velocity ω, and the soap starts to slide toward the downhill edge. Show that the equation of motion for the soap has the form $\ddot{x} - \omega^2 x = -g\sin\omega t$, where x is the soap's distance from the downhill edge. Solve this for $x(t)$, given that $x(0) = x_o$. [You'll need to use the method used to solve Equation (5.48). You can easily solve the homogeneous equation; for a particular solution try $x = A\sin\omega t$ and solve for A.]

The soap's height above the table is $y = x\sin(\omega t)$, and the velocity has component $\dot{x}$ up the slope and $x\omega$ normal to the slope. Therefore,

$$\mathcal{L} = \tfrac{1}{2}mv^2 - mgy = \tfrac{1}{2}m(\dot{x}^2 + x^2\omega^2) - mgx\sin(\omega t)$$

and the Lagrange equation is

$$\ddot{x} = \omega^2 x - g\sin(\omega t).$$

The homogeneous equation $\ddot{x} = \omega^2 x$ has as its general solution $x_h = C\cosh(\omega t) + D\sinh(\omega t)$. If we substitute the suggested particular solution $x = A\sin(\omega t)$ into the equation of motion, we find that the equation is satisfied if $A = g/2\omega^2$. Therefore the general solution is $x(t) = (g/2\omega^2)\sin(\omega t) + C\cosh(\omega t) + D\sinh(\omega t)$. Using the initial conditions that $x(0) = x_o$ and $\dot{x}(0) = 0$, we see that $C = x_o$ and $D = -g/2\omega^2$, so the solution is

$$x(t) = x_o\cosh(\omega t) + (g/2\omega^2)[\sin(\omega t) - \sinh(\omega t)].$$

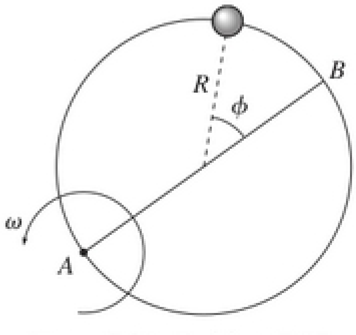


Figure 7.16 Problem 7.35

7.35 ⋆⋆ Figure 7.16 is a bird's-eye view of a smooth horizontal wire hoop that is forced to rotate at a fixed angular velocity ω about a vertical axis through the point A. A bead of mass m is threaded on the hoop and is free to move around it, with its position specified by the angle ϕ that it makes at the center with the diameter AB. Find the Lagrangian for this system using ϕ as your generalized coordinate. (Read the hint in Problem 7.29.) Use the Lagrange equation of motion to show that the bead oscillates about the point B exactly like a simple pendulum. What is the frequency of these oscillations if their amplitude is small?

Let O be the center of the hoop. The line AO is turning with angular velocity ω, so the speed of O relative to A is $v_o = R\omega$. The line from O to the bead is turning with angular velocity $\dot{\phi} + \omega$, so the speed of the bead relative to O is $v_b = R(\dot{\phi} + \omega)$. The velocity of the bead relative to A is $\mathbf{v} = \mathbf{v}_b + \mathbf{v}_o$, and

$$T = \tfrac{1}{2}m\mathbf{v}^2 = \tfrac{1}{2}m(\mathbf{v}_b^2 + \mathbf{v}_o^2 + 2\mathbf{v}_b \cdot \mathbf{v}_o) = \tfrac{1}{2}mR^2[(\dot{\phi} + \omega)^2 + \omega^2 + 2(\dot{\phi} + \omega)\omega\cos\phi]$$

since ϕ is the angle between $\mathbf{v}_b$ and $\mathbf{v}_o$. Because the hoop is horizontal, the potential energy is constant, and we may as well take it to be $U = 0$, so that $\mathcal{L} = T$. The Lagrange equation is

$$\frac{\partial T}{\partial \phi} = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} \quad \text{or} \quad -(\dot{\phi} + \omega)\omega\sin\phi = \frac{d}{dt}[(\dot{\phi} + \omega) + \omega\cos\phi] = (\ddot{\phi} - \dot{\phi}\omega\sin\phi),$$

where I have cancelled a common factor of mR^2. The first term on the left cancels the second on the right, and we're left with $\ddot{\phi} = -\omega^2\sin\phi$, which is exactly the equation of a simple pendulum with g/L replaced by ω^2, oscillating about the point $\phi = 0$ (that is, B). The frequency of small oscillations is evidently just ω.

7.37 ★★★ Two equal masses, $m_1 = m_2 = m$, are joined by a massless string of length L that passes through a hole in a frictionless horizontal table. The first mass slides on the table while the second hangs below the table and moves up and down in a vertical line. **(a)** Assuming the string remains taut, write down the Lagrangian for the system in terms of the polar coordinates (r, ϕ) of the mass on the table. **(b)** Find the two Lagrange equations and interpret the ϕ equation in terms of the angular momentum ℓ of the first mass. **(c)** Express $\dot{\phi}$ in terms of ℓ and eliminate $\dot{\phi}$ from the r equation. Now use the r equation to find the value $r = r_o$ at which the first mass can move in a circular path. Interpret your answer in Newtonian terms. **(d)** Suppose the first mass is moving in this circular path and is given a small radial nudge. Write $r(t) = r_o + \epsilon(t)$ and rewrite the r equation in terms of $\epsilon(t)$ dropping all powers of $\epsilon(t)$ higher than linear. Show that the circular path is stable and that $r(t)$ oscillates sinusoidally about r_o. What is the frequency of its oscillations?

(a) The hanging mass is a distance $L - r$ below the table. Thus it's KE is $\frac{1}{2}m\dot{r}^2$ and its PE is $-mg(L - r)$, or just mgr, if we drop an uninteresting constant. The mass on the table has KE = $\frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$ and PE which is constant and we may as well take to be zero. Thus

$$\mathcal{L} = m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\phi}^2 - mgr.$$

(b) The two Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad \text{or} \quad mr\dot{\phi}^2 - mg = 2m\ddot{r}$$

and

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad \text{or} \quad 0 = \frac{d}{dt}(mr^2\dot{\phi}).$$

The ϕ equation says simply that the angular momentum $\ell = mr^2\dot{\phi}$ is constant.
(c) Clearly $\dot{\phi} = \ell/mr^2$, and the r equation can be rewritten as

$$2m\ddot{r} = \frac{\ell^2}{mr^3} - mg. \tag{S7.3}$$

The length r can remain constant if and only if $\ddot{r} = 0$. This requires that $\ell^2/mr_o^{\,3} = mg$. (This condition says that the centripetal force needed to keep the upper mass in a circular path must equal the tension needed to hold the lower mass at a fixed height.) Therefore, $r_o = [\ell^2/(m^2 g)]^{1/3}$
(d) If $r = r_o + \epsilon$, then (S7.3) becomes

$$2m\ddot{\epsilon} = \frac{\ell^2}{m(r_o + \epsilon)^3} - mg \; = \; \frac{\ell^2}{mr_o^3}\left(1 + \frac{\epsilon}{r_o}\right)^{-3} - mg \; \approx \; \frac{\ell^2}{mr_o^3}\left(1 - 3\frac{\epsilon}{r_o}\right) - mg$$

where, to get the final expression, I used the binomial approximation. The first and last terms in the final expression cancel, and we are left with $2m\ddot{\epsilon} = -3(\ell^2/mr_o^{\,4})\epsilon$, which implies that ϵ oscillates sinusoidally with frequency $\sqrt{3/2}\,\ell/mr_o^{\,2} = \sqrt{3g/2r_o}$, since $\ell/m = \sqrt{gr_o^3}$. In particular, since the displacement ϵ oscillates, the equilibrium at $r = r_o$ is stable.

7.39 ⋆⋆⋆ **(a)** Write down the Lagrangian for a particle moving in three dimensions under the influence of a conservative central force with potential energy $U(r)$, using spherical polar coordinates (r, θ, ϕ). **(b)** Write down the three Lagrange equations and explain their significance in terms of radial acceleration, angular momentum, and so forth. (The θ equation is the tricky one, since you will find it implies that the ϕ component of $\boldsymbol{\ell}$ varies with time, which seems to contradict conservation of angular momentum. Remember, however, that ℓ_ϕ is the component of $\boldsymbol{\ell}$ in a *variable* direction.) **(c)** Suppose that initially the motion is in the equatorial plane (that is, $\theta_o = \pi/2$ and $\dot{\theta}_o = 0$). Describe the subsequent motion. **(d)** Suppose instead that the initial motion is along a line of longitude (that is, $\dot{\phi}_o = 0$). Describe the subsequent motion.

(a) The particle's velocity has spherical polar components

$$\mathbf{v} = (\dot{r}, r\dot{\theta}, r\sin\theta\,\dot{\phi}) \tag{S7.4}$$

Therefore

$$\mathcal{L} = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\!\theta\,\dot{\phi}^2\right) - U(r)$$

(b) The r, θ, and ϕ equations are

$$m\ddot{r} = mr\left(\dot{\theta}^2 + \sin^2\!\theta\,\dot{\phi}^2\right) - \partial U/\partial r \tag{S7.5}$$

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = mr^2\sin\theta\cos\theta\,\dot{\phi}^2 \tag{S7.6}$$

$$\frac{d}{dt}\left(mr^2\sin^2\!\theta\,\dot{\phi}\right) = 0 \tag{S7.7}$$

The first term on the right of the r equation (S7.5) can be recognized as mv_{tg}^2/r, where v_{tg} is the tangential velocity [obtained from (S7.4) by dropping the first component]. This is just the centrifugal force F_{cf}, so the radial equation is just $m\ddot{r} = F_{\text{cf}} + F_r$.

The term in parentheses in the ϕ equation (S7.7) can be recognized as ℓ_z. Therefore the ϕ equation states simply that ℓ_z is constant, as expected.

The term in parentheses in the θ equation (S7.6) can be recognized as ℓ_ϕ. (To see this, just evaluate $\boldsymbol{\ell} = m\mathbf{r} \times \mathbf{v}$ in polar coordinates.) Thus the θ equation gives the rate of change of ℓ_ϕ. This result is surprising at first, since we know that the vector $\boldsymbol{\ell}$ is constant. However, as the particle moves, the vector $\hat{\boldsymbol{\phi}}$ changes; thus, ℓ_ϕ is the component of a fixed vector in a *variable* direction and does in fact change, as in (S7.6).

(c) If initially $\theta = \pi/2$ and $\dot{\theta} = 0$, then the θ equation (S7.6) shows that the product $mr^2\dot{\theta}$ remains constant. Thus $\dot{\theta}$ remains equal to zero and θ remains equal to $\pi/2$. That is, the motion remains in the equatorial plane $\theta = \pi/2$, consistent with our knowledge that the motion is confined to a plane.
(d) If initially $\dot{\phi} = 0$, then the ϕ equation (S7.7) shows that the product $mr^2 \sin^2\theta\, \dot{\phi}$ remains zero. Thus $\dot{\phi}$ remains zero and ϕ is constant. Therefore, the motion is confined to the longitudinal plane $\phi = \phi_o$.

7.41 ⋆⋆⋆ Consider a bead of mass m sliding without friction on a wire that is bent in the shape of a parabola and is being spun with constant angular velocity ω about its vertical axis, as shown in Figure 7.17. Use cylindrical polar coordinates and let the equation of the parabola be $z = k\rho^2$. Write down the Lagrangian in terms of ρ as the generalized coordinate. Find the equation of motion of the bead and determine whether there are positions of equilibrium, that is, values of ρ at which the bead can remain fixed, without sliding up or down the spinning wire. Discuss the stability of any equilibrium positions you find.

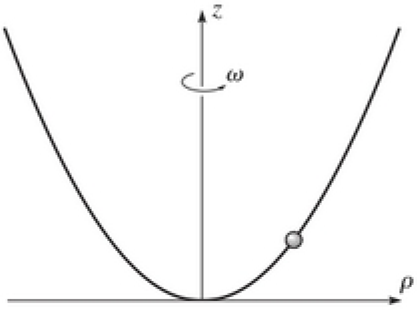


Figure 7.17 Problem 7.41

The potential energy is just $U = mgz = mgk\rho^2$. The kinetic energy is

$$T = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\left(\dot{\rho}^2 + (\rho\dot{\phi})^2 + \dot{z}^2\right) = \tfrac{1}{2}m\left(\dot{\rho}^2 + \rho^2\omega^2 + 4k^2\rho^2\dot{\rho}^2\right).$$

Therefore the Lagrangian is

$$\mathcal{L} = T - U = \tfrac{1}{2}m\left(\dot{\rho}^2 + \rho^2\omega^2 + 4k^2\rho^2\dot{\rho}^2\right) - mgk\rho^2,$$

and the equation of motion is (after a little algebra)

$$(1 + 4k^2\rho^2)\ddot{\rho} + 4k^2\rho\dot{\rho}^2 = (\omega^2 - 2gk)\rho. \tag{S7.8}$$

A position ρ_o is an equilibrium point if placing the bead at ρ_o with $\dot{\rho} = 0$ ensures that $\ddot{\rho} = 0$ (guaranteeing that $\dot{\rho}$ remains zero and ρ constant). According to Eq. (S7.8) this will be true if and only if

$$(\omega^2 - 2gk)\rho = 0. \tag{S7.9}$$

This can be satisfied in two ways: First the system is in equilibrium if $\rho = 0$, that is, if the bead is exactly at the bottom of the wire. To decide whether this equilibrium is stable, we have only to imagine pulling the bead a small distance to one side. With ρ and $\dot{\rho}$ very small Eq. (S7.8) implies that

$$\ddot{\rho} \approx (\omega^2 - 2gk)\rho.$$

If $\omega^2 < 2gk$, the term in parentheses is negative, and the bead accelerates back to the bottom, so the equilibrium is stable. If $\omega^2 > 2gk$, the term in parentheses is positive, and the bead accelerates away from the bottom, so the equilibrium is unstable.

If $\omega^2 = 2gk$, the condition (S7.9) is satisfied for *any* value of ρ. That is, with $\omega = \sqrt{2gk}$, the bead will be in equilibrium anywhere on the wire. To investigate the stability of this equilibrium, we imagine giving the bead a small nudge. From Eq. (S7.8), with the right side equal to zero, we see that $\ddot{\rho}$ is always negative for any positive ρ. Thus if we nudge the bead outward, its negative acceleration will slow it down, though not actually stop it. If we nudge it inward it will actually speed up. Either way it will not return to its original position.

7.43 ⋆⋆⋆ [Computer] Consider a massless wheel of radius R mounted on a frictionless horizontal axis. A point mass M is glued to the edge, and a massless string is wrapped several times around the perimeter and hangs vertically down with a mass m suspended from its bottom end. (See Figure 4.28.) Initially I am holding the wheel with M vertically below the axle. At $t = 0$, I release the wheel, and m starts to fall vertically down. **(a)** Write down the Lagrangian $\mathcal{L} = T - U$ as a function of the angle ϕ through which the wheel has turned. Find the equation of motion and show that, provided $m < M$, there is one position of stable equilibrium. **(b)** Assuming $m < M$, sketch the potential energy $U(\phi)$ for $-\pi \le \phi \le 4\pi$ and use your graph to explain the equilibrium positions you found. **(c)** Because the equation of motion cannot be solved in terms of elementary functions, you are going to solve it numerically. This requires that you choose numerical values for the various parameters. Take $M = g = R = 1$ (this amounts to a convenient choice of units) and $m = 0.7$. Before solving the equation make a careful plot of $U(\phi)$ against ϕ and predict the kind of motion expected when M is released from rest at $\phi = 0$. Now solve the equation of motion for $0 \le t \le 20$ and verify your prediction. **(d)** Repeat part (c), but with $m = 0.8$.

(a) $\mathcal{L} = \frac{1}{2}(M + m)R^2\dot{\phi}^2 - MgR(1 - \cos\phi) + mgR\phi$, and the equation of motion is

$$(M + m)R\ddot{\phi} = -Mg\sin\phi + mg. \tag{S7.10}$$

Equilibrium positions require that $\ddot{\phi} = 0$ or $\sin\phi = m/M$. Provided $m < M$, this has two solutions, $\phi = \arcsin(m/M)$, one with $0 < \phi < \pi/2$ and one with $\pi/2 < \phi < \pi$. If the wheel turns a little away from the first of these, it is easy to see from (S7.10) that it accelerates back and the equilibrium is stable. In the same way, you can see that the second is unstable.

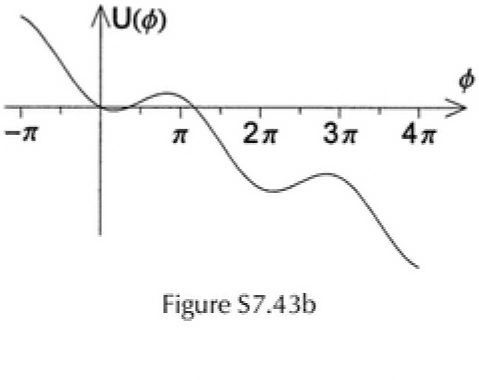


Figure S7.43b

(b) The stable equilibrium is the minimum at about $\pi/6$; the unstable is the maximum at about $5\pi/6$. Notice that there is actually a whole sequence of equilibriums, spaced at intervals of 2π (that is, differing by one or more complete revolutions).

(c) Based on the left-hand plot of $U(\phi)$ against ϕ in Figure S7.43c, it is clear that when released from $\phi = 0$, the mass M will swing out to an angle a bit bigger than $\pi/2$, swing back to $\phi = 0$, and continue to oscillate indefinitely, as confirmed by the right-hand picture of $\phi(t)$ against t.

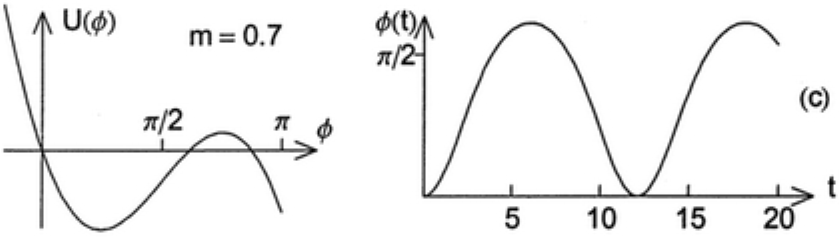


Figure S7.43c

(d) With $m = 0.8$, the maximum in $U(\phi)$ just beyond $\phi = \pi/2$ is not high enough to stop the wheel, which continues to roll indefinitely. You can see in the right-hand picture that, by the time $t = 20$, it has made more than 9 complete revolutions. (Note the very different vertical scale.)

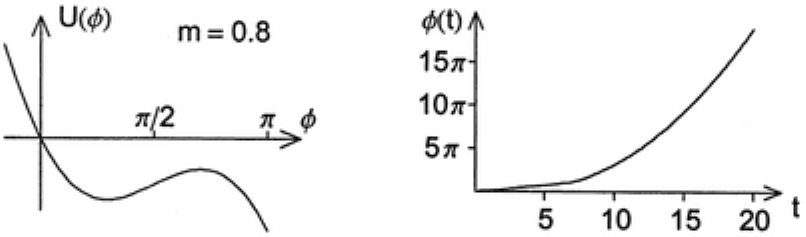


Figure S7.43d

SECTION 7.8 More About Conservation Laws

7.45 ⋆⋆ **(a)** Verify that the coefficients A_{ij} in the important expression (7.94) for the kinetic energy of any "natural" system are symmetric; that is, $A_{ij} = A_{ji}$. **(b)** Prove that for any n variables $v_1, \cdots, v_n$

$$\frac{\partial}{\partial v_i}\sum_{j,k} A_{jk}v_j v_k = 2\sum_j A_{ij}v_j \, .$$

[*Hint:* Start with the case that $n = 2$, for which you can write out the sums in full. Notice that you need the result of part (a).] This identity is useful in many areas of physics; we needed it to prove the expression (7.96) for the generalized momentum p_i.

(a) If you look at the definition (7.95) of A_{jk}, you will see that A_{jk} differs from A_{kj} only in the order of the two terms in the scalar product. Since the scalar product is commutative, these two expressions are equal.

(b) If we consider first the case of just two variables, the sum in question is

$$S = \sum_{j,k} A_{jk}v_j v_k = A_{11}v_1^2 + A_{12}v_1v_2 + A_{21}v_2v_1 + A_{22}v_2^2$$

$$= A_{11}v_1^2 + 2A_{12}v_1v_2 + A_{22}v_2^2$$

where, in the second line, I used the fact that $A_{12} = A_{21}$. Differentiating with respect to v_1, we find that $\partial S/\partial v_1 = 2A_{11}v_1 + 2A_{12}v_2 = 2\sum_j A_{1j}v_j$, which is the claimed result for $i = 1$. The case $i = 2$ works in the same way.

If there are n variables, then, before differentiating with respect to v_i, it helps to separate out the terms that depend on v_i from those that do not:

$$S = \sum_{j,k} A_{jk} v_j v_k = A_{ii} v_i^2 + \sum_{k \neq i} A_{ik} v_i v_k + \sum_{j \neq i} A_{ji} v_j v_i + \text{terms not involving } v_i$$

$$= A_{ii} v_i^2 + 2 \sum_{j \neq i} A_{ij} v_i v_j + \text{terms not involving } v_i.$$

Here, in passing to the second line, I replaced the dummy index k by j in the sum $\sum_{k \neq i}$, and used the fact that $A_{ji} = A_{ij}$ in the sum $\sum_{j \neq i}$. Differentiating with respect to v_i we find that

$$\frac{\partial S}{\partial v_i} = 2A_{ii} v_i + 2 \sum_{j \neq i} A_{ij} v_j = 2 \sum_j A_{ij} v_j.$$

7.47 ⋆⋆⋆ In Chapter 4 (at the end of Section 4.7) I claimed that, for a system with one degree of freedom, positions of stable equilibrium "normally" correspond to minima of the potential energy $U(q)$. Using Lagrangian mechanics, you can now prove this claim. **(a)** Consider a one-degree system of N particles with positions $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q)$, where q is the one generalized coordinate and the transformation between $\mathbf{r}$ and q does not depend on time; that is, q is what we have now agreed to call "natural." (This is the meaning of the qualification "normally" in the statement of the claim. If the transformation depends on time, then the claim is not necessarily true.) Prove that the KE has the form $T = \frac{1}{2} A \dot{q}^2$, where $A = A(q) > 0$ may depend on q but not on $\dot{q}$. [This corresponds exactly to the result (7.94) for n degrees of freedom. If you have trouble with the proof here, review the proof there.] Show that the Lagrange equation of motion has the form

$$A(q)\ddot{q} = -\frac{dU}{dq} - \frac{1}{2}\frac{dA}{dq}\dot{q}^2.$$

(b) A point q_o is an equilibrium point if, when the system is placed at q_o with $\dot{q} = 0$, it remains there. Show that q_o is an equilibrium point if and only if $dU/dq = 0$. **(c)** Show that the equilibrium is stable if and only if U is minimum at q_o. **(d)** If you did Problem 7.30, show that the pendulum of that problem does not satisfy the conditions of this problem and that the result proved here is false for that system.

(a) The KE is $T = \frac{1}{2} \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha^2$ and, because $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q)$, it follows that $\dot{\mathbf{r}}_\alpha = \dot{q}(d\mathbf{r}_\alpha/dq)$. Therefore

$$T = \tfrac{1}{2} \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha^2 = \tfrac{1}{2}\dot{q}^2 \sum_\alpha m_\alpha \frac{d\mathbf{r}_\alpha}{dq} \cdot \frac{d\mathbf{r}_\alpha}{dq} = \tfrac{1}{2} A(q)\dot{q}^2 \quad \text{where} \quad A(q) = \sum_\alpha m_\alpha \frac{d\mathbf{r}_\alpha}{dq} \cdot \frac{d\mathbf{r}_\alpha}{dq} > 0$$

Because $\mathcal{L} = T - U = \frac{1}{2} A(q)\dot{q}^2 - U(q)$, the two derivatives of $\mathcal{L}$ are

$$\frac{\partial \mathcal{L}}{\partial q} = \tfrac{1}{2} A'(q)\dot{q}^2 - U'(q) \qquad \text{and} \qquad \frac{\partial \mathcal{L}}{\partial \dot{q}} = A(q)\dot{q}$$

and the Lagrange equation is (after a little algebra) $A(q)\ddot{q} = -U'(q) - \frac{1}{2} A'(q)\dot{q}^2$.

(b) If q_o is an equilibrium point and if the system is placed at q_o with $\dot{q} = 0$, then the system stays put, so $\ddot{q} = 0$. By the Lagrange equation this implies that $U'(q_o)$ must be 0. Conversely, if $U'(q_o) = 0$, then when the system is placed at q_o with $\dot{q} = 0$, $\ddot{q} = 0$ and the system stays put.

(c) Suppose that q_o is stable and the system is placed at rest close to q_o at $q_o + \epsilon$ with $\epsilon > 0$. Because q_o is stable, the system accelerates back toward q_o, so $\ddot{q}$ must be negative. Inspecting the Lagrange equation, we see that U' must be positive, which implies that U is at a minimum at q_o. Running the argument backward, we see that if U is minimum, then the equilibrium is stable.

(d) In Problem 7.30, the relation between the position $\mathbf{r}$ of the pendulum and the generalized coordinate ϕ is time-*dependent*. The PE is minimum at $\phi = 0$, whereas the equilibrium occurs at $\phi = -\beta \neq 0$.

SECTION 7.9 Lagrange's Equations for Magnetic Forces

7.49 ⋆⋆ Consider a particle of mass m and charge q moving in a uniform constant magnetic field $\mathbf{B}$ in the z direction. **(a)** Prove that $\mathbf{B}$ can be written as $\mathbf{B} = \nabla \times \mathbf{A}$ with $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$. Prove equivalently that in cylindrical polar coordinates, $\mathbf{A} = \frac{1}{2}B\rho\,\hat{\boldsymbol{\phi}}$. **(b)** Write the Lagrangian (7.103) in cylindrical polar coordinates and find the three corresponding Lagrange equations. **(c)** Describe in detail those solutions of the Lagrange equations in which ρ is a constant.

(a) If $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r} = \frac{1}{2}(B_y z - B_z y,\, B_z x - B_x z,\, B_x y - B_y x)$, then

$$(\nabla \times \mathbf{A})_x = \partial_y A_z - \partial_z A_y = B_x.$$

(Remember that $\mathbf{B}$ is uniform and constant.) Since the y and z components work the same way, we conclude that $\mathbf{B} = \nabla \times \mathbf{A}$. In polar coordinates, $\mathbf{B} = B\hat{\mathbf{z}}$ and $\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}$, so

$$\mathbf{A} = \tfrac{1}{2}\mathbf{B} \times \mathbf{r} = \tfrac{1}{2}B\hat{\mathbf{z}} \times (\rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}) = \tfrac{1}{2}B\rho\hat{\boldsymbol{\phi}}$$

since $\hat{\mathbf{z}} \times \hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\phi}}$.

(b) Since there is no electric field, $V = 0$, and since $\dot{\mathbf{r}} = \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\mathbf{z}}$,

$$\mathcal{L} = \tfrac{1}{2}m\dot{\mathbf{r}}^2 + q\dot{\mathbf{r}} \cdot \mathbf{A} = \tfrac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) + \tfrac{1}{2}qB\rho^2\dot{\phi}.$$

The three Lagrange equations are

$$m\ddot{\rho} = m\rho\dot{\phi}^2 + qB\rho\dot{\phi}, \qquad \frac{d}{dt}\left(m\rho^2\dot{\phi} + \tfrac{1}{2}qB\rho^2\right) = 0, \qquad \text{and} \qquad m\ddot{z} = 0.$$

(c) In any case, the solution of the z equation is $z = z_o + v_{zo}t$; that is, the particle moves uniformly in the direction of $\mathbf{B}$. If $\rho =$ constant, the ρ equation reduces to $m\dot{\phi}^2 + qB\dot{\phi} = 0$. Therefore, either $\dot{\phi} = 0$ (in which case the particle moves straight along a field line) or $\dot{\phi} = -qB/m$. In this second case, the particle moves clockwise around the z axis (assuming q is positive) at the same time it moves in the z direction with constant velocity; this results in the helical motion described in Section 2.7, with angular velocity equal to the cyclotron freqency $\omega = qB/m$.

SECTION 7.10 Lagrange Multipliers and Constraint Forces

7.51 ⋆ Write down the Lagrangian for the simple pendulum of Figure 7.2 in terms of the rectangular coordinates x and y. These coordinates are constrained to satisfy the constraint equation $f(x, y) = \sqrt{x^2 + y^2} = l$. **(a)** Write down the two modified Lagrange equations (7.118) and (7.119). Comparing these with the two components of Newton's second law, show that the Lagrange multiplier is (minus) the tension in the rod. Verify Equation (7.122) and the corresponding equation in y. **(b)** The constraint equation can be written in many different ways. For example we could have written $f'(x, y) = x^2 + y^2 = l^2$. Check that using this function would have given the same physical results.

$\mathcal{L}(x, y) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy$.

(a) With the constraint $f(x, y) = \sqrt{x^2 + y^2} = l$, the two modified Lagrange equations are

$$\lambda\frac{x}{l} = m\ddot{x} \qquad \text{and} \qquad mg + \lambda\frac{y}{l} = m\ddot{y}.$$

If F^{t} denotes the tension in the rod, the two Newtonian equations are:

$$-F^{\text{t}}\sin\phi = -F^{\text{t}}\frac{x}{l} = m\ddot{x} \qquad \text{and} \qquad mg - F^{\text{t}}\cos\phi = mg - F^{\text{t}}\frac{y}{l} = m\ddot{y}. \tag{S7.11}$$

Comparing corresponding equations we see that $\lambda = -F^{\mathrm{t}}$. Perhaps more important, (7.122) is satisfied for both x and y components:

$$\lambda \frac{\partial f}{\partial x} = \lambda \frac{x}{l} = -F^{\mathrm{t}} \sin\phi = F^{\mathrm{t}}_x$$

and likewise for the y component.

(b) If instead we use the constraint equation $f'(x, y) = x^2 + y^2 = l^2$, the two modified Lagrange equations are

$$\lambda' 2x = m\ddot{x} \qquad \text{and} \qquad mg + \lambda' 2y = m\ddot{y}.$$

Comparing with the two Newtonian equations (S7.11), we find that in this case $\lambda' = -F^{\mathrm{t}}/2l$, but again (7.122) is satisfied:

$$\lambda' \frac{\partial f'}{\partial x} = \lambda'(2x) = -\frac{F^{\mathrm{t}}}{2l}(2l\cos\phi) = F^{\mathrm{t}}_x$$

and likewise for the y component.

CHAPTER 8

Two-Body Central-Force Problems

PROBLEMS AND SOLUTIONS

SECTION 8.2 CM and Relative Coordinates; Reduced Mass

8.1 ⋆ Verify that the positions of two particles can be written in terms of the CM and relative positions as $\mathbf{r}_1 = \mathbf{R} + m_2\mathbf{r}/M$ and $\mathbf{r}_2 = \mathbf{R} - m_1\mathbf{r}/M$. Hence confirm that the total KE of the two particles can be expressed as $T = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2$, where μ denotes the reduced mass $\mu = m_1m_2/M$.

You can regard Eqs. (8.4) and (8.7) as two simultaneous equations to be solved for $\mathbf{r}_1$ and $\mathbf{r}_2$. If you multiply (8.4) by m_2/M and add to (8.7), this gives $\mathbf{r}_1 = \mathbf{R} + (m_2/M)\mathbf{r}$. Similarly, if you multiply (8.4) by m_1/M and subtract from (8.7), this gives $\mathbf{r}_2 = \mathbf{R} - (m_1/M)\mathbf{r}$. If you differentiate these results with respect to t, the required expression for T follows exactly as in (8.10).

8.2 ⋆⋆ Although the main topic of this chapter is the motion of two particles subject to no external forces, many of the ideas [for example, the splitting of the Lagrangian $\mathcal{L}$ into two independent pieces $\mathcal{L} = \mathcal{L}_{\text{cm}} + \mathcal{L}_{\text{rel}}$ as in Equation (8.13)] extend easily to more general situations. To illustrate this, consider the following: Two masses m_1 and m_2 move in a uniform gravitational field $\mathbf{g}$ and interact via a potential energy $U(r)$. **(a)** Show that the Lagrangian can be decomposed as in (8.13). **(b)** Write down Lagrange's equations for the three CM coordinates X, Y, Z and describe the motion of the CM. Write down the three Lagrange equations for the relative coordinates and show clearly that the motion of $\mathbf{r}$ is the same as that of a single particle of mass equal to the reduced mass μ, with position $\mathbf{r}$ and potential energy $U(r)$.

8.3 ⋆⋆ Two particles of masses m_1 and m_2 are joined by a massless spring of natural length L and force constant k. Initially I am holding both masses (one in each hand), with m_1 a distance L vertically above m_2. At $t = 0$, I throw m_1 vertically up with speed v_o, at the same time releasing m_2 (at rest). Find the positions of the two masses at any subsequent time t and describe the motion. [*Hints*: See Problem 8.2. Assume that v_o is small enough that the two masses never collide.]

The motion is obviously confined to a vertical line, so we can treat it as a one-dimensional problem with coordinates y_1 and y_2 measured vertically up from the initial position of m_2. The Lagrangian is $\mathcal{L} = \frac{1}{2}M\dot{Y}^2 - MgY + \frac{1}{2}\mu\dot{y}^2 - \frac{1}{2}k(y - L)^2$, where M and μ are the total and reduced masses, and Y and y are the CM and relative positions. The two Lagrange equations are

$$\ddot{Y} = -g \qquad \text{and} \qquad \mu\ddot{y} = -k(y - L),$$

with solutions $Y = Y_o + \dot{Y}_o t - \frac{1}{2}gt^2$ (where $Y_o = Lm_1/M$ and $\dot{Y}_o = v_o m_1/M$) and $y = L + A\sin\omega t$ (where $\omega = \sqrt{k/\mu}$ and $A = v_o/\omega$). That is, the CM moves up then down like a body in free fall, while the relative position oscillates in SHM. Using Eq. (8.9), we can find the individual positions

$$y_1 = L + \frac{m_1}{M}v_o t - \tfrac{1}{2}gt^2 + \frac{m_2 v_o}{M\omega}\sin\omega t \qquad \text{and} \qquad y_2 = \frac{m_1}{M}v_o t - \tfrac{1}{2}gt^2 - \frac{m_1 v_o}{M\omega}\sin\omega t.$$

SECTION 8.3 The Equations of Motion

8.5 ⋆ The momentum $\mathbf{p}$ conjugate to the relative position $\mathbf{r}$ is defined with components $p_x = \partial\mathcal{L}/\partial\dot{x}$ and so on. Prove that $\mathbf{p} = \mu\dot{\mathbf{r}}$. Prove also that in the CM frame, $\mathbf{p}$ is the same as $\mathbf{p}_1$ the momentum of particle 1 (and also $-\mathbf{p}_2$).

Because $\mathcal{L} = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r)$, we find $p_x = \partial\mathcal{L}/\partial\dot{x} = \mu\dot{x}$. Combining this with the corresponding results for the y and z components, we find $\mathbf{p} = \mu\dot{\mathbf{r}}$.

Since $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, the last result implies that

$$\mathbf{p} = \frac{m_1 m_2}{m_1 + m_2}(\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2) = \frac{1}{m_1 + m_2}(m_2\mathbf{p}_1 - m_1\mathbf{p}_2).$$

This last result is true in any frame, but in the CM frame, $\mathbf{p}_2 = -\mathbf{p}_1$, so the result becomes $\mathbf{p} = \mathbf{p}_1$. In the CM frame, the relative momentum is the same as $\mathbf{p}_1$.

8.7 ⋆⋆ **(a)** Using elementary Newtonian mechanics find the period of a mass m_1 in a circular orbit of radius r around a *fixed* mass m_2. **(b)** Using the separation into CM and relative motions, find the corresponding period for the case that m_2 is not fixed and the masses circle each other a constant distance r apart. Discuss the limit of this result if $m_2 \to \infty$. **(c)** What would be the orbital period if the earth were replaced by a star of mass equal to the solar mass, in a circular orbit, with the distance between the sun and star equal to the present earth–sun distance? (The mass of the sun is more than 300,000 times that of the earth.)

(a) Newton's second law in the form $m\mathbf{a} = \mathbf{F}$, with $\mathbf{a}$ equal to the centripetal acceleration (magnitude $\omega^2 r$), implies that $m_1\omega^2 r = Gm_1m_2/r^2$. Substituting $\omega = 2\pi/\tau$ and solving for the period τ, we find $\tau = 2\pi r^{3/2}/\sqrt{Gm_2}$. (Note how the mass m_1 cancels out.)
(b) The equation of motion, $\mu\ddot{\mathbf{r}} = \mathbf{F}$, for the relative motion (with μ equal to the *reduced* mass m_1m_2/M) implies that $\mu\omega^2 r = Gm_1m_2/r^2$. This is the same as before, except that the mass on the left is now the reduced mass μ instead of m_1. This no longer cancels the factor m_1 on the right, so the final answer contains m_1m_2/μ instead of m_2:

$$\tau = \frac{2\pi r^{3/2}}{\sqrt{Gm_1m_2/\mu}} = \frac{2\pi r^{3/2}}{\sqrt{GM}}. \tag{S8.1}$$

If $m_2 \gg m_1$, then $M \approx m_2$ and (S8.1) approaches the answer of part (a).
(c) If $m_1 = m_2$, then $M = 2m_2$, and the period (S8.1) is $1/\sqrt{2}$ times that in part (a), or, for the case at hand, 0.71 years.

8.9 ⋆⋆ Consider two particles of equal masses, $m_1 = m_2$, attached to each other by a light straight spring (force constant k, natural length L) and free to slide over a frictionless horizontal table. **(a)** Write down the Lagrangian in terms of the coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$, and rewrite it in terms of the CM and relative positions, $\mathbf{R}$ and $\mathbf{r}$, using polar coordinates (r, ϕ) for $\mathbf{r}$. **(b)** Write down and solve the Lagrange equations for the CM coordinates X, Y. **(c)** Write down the Lagrange equations for r and ϕ. Solve these for the two special cases that r remains constant and that ϕ remains constant. Describe the corresponding motions. In particular, show that the frequency of oscillations in the second case is $\omega = \sqrt{2k/m_1}$.

(a)

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}m(\dot{\mathbf{r}}_1^2 + \dot{\mathbf{r}}_2^2) - \tfrac{1}{2}k\left(|\mathbf{r}_1 - \mathbf{r}_2| - L\right)^2 \\ &= \tfrac{1}{2}M\dot{\mathbf{R}}^2 + \tfrac{1}{2}\mu\dot{\mathbf{r}}^2 - \tfrac{1}{2}k(r - L)^2 \\ &= \tfrac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \tfrac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - \tfrac{1}{2}k(r - L)^2\end{aligned}$$

(b) The equations are $M\ddot{X} = 0$ and $M\ddot{Y} = 0$, with solutions $\mathbf{R} = \mathbf{R}_\mathrm{o} + \dot{\mathbf{R}}_\mathrm{o}t$. That is, the CM moves like a free particle.

(c) The r and ϕ equations are

$$\mu\ddot{r} = \mu r\dot{\phi}^2 - k(r - L) \qquad \text{and} \qquad \mu r^2\dot{\phi} = \text{const.}$$

If $r = \text{const}$, the ϕ equation tells us that $\dot{\phi} = \text{const} = \omega$, say, and the r equation becomes $k(r - L) = \mu r\omega^2$, which says that the spring force supplies the centripetal acceleration $r\omega^2$. The two particles circle each other at fixed radius.

If ϕ is constant, the r equation becomes $\mu\ddot{r} = -k(r - L)$, which implies that $r = L + A\cos(\omega t - \delta)$, where $\omega = \sqrt{k/\mu} = \sqrt{2k/m_1}$. The arrangement doesn't rotate, but undergoes radial oscillations with angular frequency $\sqrt{2k/m_1}$.

8.11 ⋆⋆ Consider two particles interacting by a Hooke's law potential energy, $U = \frac{1}{2}kr^2$, where $\mathbf{r}$ is their relative position $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, and subject to no external forces. Show that $\mathbf{r}(t)$ describes an ellipse. Hence show that both particles move on similar ellipses around their common CM. [This is surprisingly awkward. Perhaps the simplest procedure is to choose the xy plane as the plane of the orbit and then solve the equation of motion (8.15) for x and y. Your solution will have the form $x = A\cos\omega t + B\sin\omega t$, with a similar expression for y. If you solve these for $\sin\omega t$ and $\cos\omega t$ and remember that $\sin^2 + \cos^2 = 1$, you can put the orbital equation in the form $ax^2 + 2bxy + cy^2 = k$ where k is a positive constant. Now invoke the standard result that if a and c are positive and $ac > b^2$, this equation defines an ellipse.]

From the Lagrangian $\mathcal{L} = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - \frac{1}{2}kr^2$ we find the equation of motion $\mu\ddot{\mathbf{r}} = -k\mathbf{r}$. Since angular momentum is conserved, the vector $\mathbf{r}$ remains in a plane [see the argument below equation (8.19)], which we can take to be the xy plane. The equation of motion then implies that

$$x = A\cos\omega t + B\sin\omega t \qquad \text{and} \qquad y = C\cos\omega t + D\sin\omega t.$$

From these there follows

$$Cx - Ay = (BC - DA)\sin\omega t \qquad \text{and} \qquad Dx - By = -(BC - DA)\cos\omega t.$$

Squaring both of these and adding, we find that $ax^2 + 2bxy + cy^2 = k$, where

$$a = C^2 + D^2, \quad b = -(CA + DB), \qquad \text{and} \qquad c = A^2 + B^2$$

and $k = (BC - DA)^2 > 0$. We are promised that this equation defines an ellipse provided a and c are positive (which they certainly are) and $ac > b^2$. So let us examine the quantity

$$ac - b^2 = (C^2 + D^2)(A^2 + B^2) - (CA + DB)^2 = (BC - DA)^2 > 0. \tag{S8.2}$$

Therefore **r** moves around an ellipse. In the CM frame, $\mathbf{r}_1 = (m_2/M)\mathbf{r}$ and $\mathbf{r}_2 = -(m_1/M)\mathbf{r}$, so both particles move around similar ellipses at opposite ends of a line through the CM. [In Eq. (S8.2), $ac - b^2$ can actually be zero. In this case, the ellipse collapses to a straight line.]

SECTION 8.4 The Equivalent One-Dimensional Problem

8.13 ★★★ Two particles whose reduced mass is μ interact via a potential energy $U = \frac{1}{2}kr^2$, where r is the distance between them.
(a) Make a sketch showing $U(r)$, the centrifugal potential energy $U_{\text{cf}}(r)$, and the effective potential energy $U_{\text{eff}}(r)$. (Treat the angular momentum ℓ as a known, fixed constant.)
(b) Find the "equilibrium" separation r_{o}, the distance at which the two particles can circle each other with constant r. [*Hint:* This requires that dU_{eff}/dr be zero.]
(c) By making a Taylor expansion of $U_{\text{eff}}(r)$ about the equilibrium point r_{o} and neglecting all terms in $(r - r_{\text{o}})^3$ and higher, find the frequency of small oscillations about the circular orbit if the particles are disturbed a little from the separation r_{o}.

(a) Since $U = \frac{1}{2}kr^2$ and $U_{\text{cf}} = \ell^2/2\mu r^2$,

$$U_{\text{eff}} = U + U_{\text{cf}} = \tfrac{1}{2}kr^2 + \frac{\ell^2}{2\mu r^2},$$

shown as the thick, gray curve.

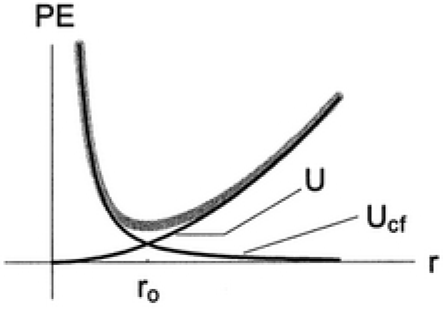


Figure S8.13

(b) We know from (8.29) that $\mu\ddot{r} = -U'_{\text{eff}}(r)$ (where the prime denotes differentiation with respect to r). Thus r can remain constant if and only if this derivative is zero. That is,

$$U'_{\text{eff}}(r_{\text{o}}) = kr_{\text{o}} - \frac{\ell^2}{\mu r_{\text{o}}^3} = 0$$

Therefore the "equilibrium" radius is $r_{\text{o}} = (\ell^2/k\mu)^{1/4}$.
(c) The Taylor expansion of $U(r)$ about r_{o} gives

$$U_{\text{eff}}(r) = U_{\text{eff}}(r_{\text{o}}) + U'_{\text{eff}}(r_{\text{o}})(r - r_{\text{o}}) + \tfrac{1}{2}U''_{\text{eff}}(r_{\text{o}})(r - r_{\text{o}})^2$$

if I ignore terms higher than quadratic. The first term on the right is constant, the second is zero [by part (b)], and the derivative in the third is easily evaluated as $U''_{\text{eff}}(r_{\text{o}}) = 4k$. If we write $r - r_{\text{o}} = \epsilon$, then the equation of motion (8.29) becomes $\mu\ddot{\epsilon} = -4k\epsilon$, and ϵ oscillates about zero with angular frequency $\omega = \sqrt{4k/\mu}$.

SECTION 8.6 The Kepler Orbits

8.15 ⋆ In deriving Kepler's third law (8.55) we made an approximation based on the fact that the sun's mass M_s is much greater than that of the planet m. Show that the law should actually read $\tau^2 = [4\pi^2/G(M_s + m)]a^3$, and hence that the "constant" of proportionality is actually a little different for different planets. Given that the mass of the heaviest planet (Jupiter) is about 2×10^{27} kg, while M_s is about 2×10^{30} kg (and some planets have masses several orders of magnitude less than Jupiter), by what percent would you expect the "constant" in Kepler's third law to vary among the planets?

Equation (8.54) (which is exact) states that $\tau^2 = ka^3$, where $k = 4\pi^2\mu/\gamma$. Because $\mu = M_s m/(M_s + m)$ and $\gamma = GM_s m$, the "constant" k is exactly $k = 4\pi^2/G(M_s + m)$. Thus k has its minimum value for Jupiter, $k_{\text{min}} = 4\pi^2/G(M_s + m_j)$, while the maximum value (for a planet of zero mass) is $k_{\text{max}} = 4\pi^2/GM_s$. The fractional difference is

$$(\text{fractional difference}) = \frac{k_{\text{max}} - k_{\text{min}}}{k_{\text{max}}} = \frac{m_j}{M_s + m_j} \approx \frac{m_j}{M_s} = 10^{-3} = 0.1\%.$$

8.17 ⋆⋆ If you did Problem 4.41 you met the **virial theorem** for a circular orbit of a particle in a central force with $U = kr^n$. Here is a more general form of the theorem that applies to any periodic orbit of a particle. **(a)** Find the time derivative of the quantity $G = \mathbf{r} \cdot \mathbf{p}$ and, by integrating from time 0 to t, show that

$$\frac{G(t) - G(0)}{t} = 2\langle T \rangle + \langle \mathbf{F} \cdot \mathbf{r} \rangle$$

where $\mathbf{F}$ is the net force on the particle and $\langle f \rangle$ denotes the average over time of any quantity f. **(b)** Explain why, if the particle's orbit is periodic and if we make t sufficiently large, we can make the left-hand side of this equation as small as we please. That is, the left side approaches zero as $t \to \infty$. **(c)** Use this result to prove that if $\mathbf{F}$ comes from the potential energy $U = kr^n$, then $\langle T \rangle = n\langle U \rangle/2$, if now $\langle f \rangle$ denotes the time average over a very long time.

(a) If $G = \mathbf{r} \cdot \mathbf{p}$, then $dG/dt = \dot{\mathbf{r}} \cdot \mathbf{p} + \dot{\mathbf{p}} \cdot \mathbf{r} = mv^2 + \mathbf{F} \cdot \mathbf{r}$. If we integrate this from 0 to t, we get $G(t) - G(0) = \int_0^t (2T + \mathbf{F} \cdot \mathbf{r})dt$, or, dividing both sides by t,

$$\frac{G(t) - G(0)}{t} = 2\langle T \rangle + \langle \mathbf{F} \cdot \mathbf{r} \rangle. \tag{S8.3}$$

(b) If the motion is periodic and if K denotes the maximum value of $|G|$ during any one cycle, then the numerator of the left side of this relation can never exceed $2K$. Therefore, as we let $t \to \infty$, the left side approaches zero.

(c) If $U = kr^n$, then $\mathbf{F} = -\nabla U = -nkr^{n-1}\hat{\mathbf{r}}$, so $\mathbf{F} \cdot \mathbf{r} = -nkr^n = -nU$. Inserting this in Eq. (S8.3) and letting $t \to \infty$, we find $0 = 2\langle T \rangle - \langle nU \rangle$, if we now understand the two angle brackets $\langle\rangle$ to denote the long-term average of whatever is between them. Therefore, $\langle T \rangle = n\langle U \rangle/2$.

8.18 ⋆⋆ An earth satellite is observed at perigee to be 250 km above the earth's surface and traveling at about 8500 m/s. Find the eccentricity of its orbit and its height above the earth at apogee. [*Hint:* The earth's radius is $R_e \approx 6.4 \times 10^6$ m. You will also need to know GM_e, but you can find this if you remember that $GM_e/R_e^2 = g$.]

8.19 ⋆⋆ The height of a satellite at perigee is 300 km above the earth's surface and it is 3000 km at apogee. Find the orbit's eccentricity. If we take the orbit to define the xy plane and the major axis in the x direction with the earth at the origin, what is the satellite's height when it crosses the y axis? [See the hint for Problem 8.18.]

We are given the satellite's heights $h_{\min} = 300$ km at perigee and $h_{\max} = 3000$ km at apogee. The corresponding distances from the earth's center are $r_{\min} = R_e + r_{\min}$ and $r_{\max} = R_e + r_{\max}$. We know from (8.50) that $r_{\min} = c/(1+\epsilon)$ and $r_{\max} = c/(1-\epsilon)$, from which it follows that $\epsilon = (r_{\max} - r_{\min})/(r_{\max} + r_{\min}) = 0.17$. From this we can find $c = (1+\epsilon)r_{\min} = 7824$ km. This is the distance from the earth's center when the satellite crosses the y axis, so the corresponding height is $h = c - R_e \approx 1400$ km.

8.20 ⋆⋆ Consider a comet which passes through its aphelion at a distance $r_{\max}$ from the sun. Imagine that, keeping $r_{\max}$ fixed, we somehow make the angular momentum ℓ smaller and smaller, though not actually zero; that is, we let $\ell \to 0$. Use equations (8.48) and (8.50) to show that in this limit the eccentricity ϵ of the elliptical orbit approaches 1 and that the distance of closest approach $r_{\min}$ approaches zero. Describe the orbit with $r_{\max}$ fixed but ℓ very small. What is the semimajor axis a?

From Eq. (8.48), $c = \ell^2/\gamma\mu$. Therefore, as $\ell \to 0$, the length $c \to 0$. From (8.50), $r_{\max} = c/(1-\epsilon)$, so, if we want to keep $r_{\max}$ fixed, ϵ must approach 1. That is, the ellipse becomes long and thin, like a cigar. Also from (8.50), $r_{\min} = c/(1+\epsilon) \to 0$, so the perihelion gets very close to the sun, as shown. Because $2a = r_{\min} + r_{\max}$, in the limit $a \to r_{\max}/2$.

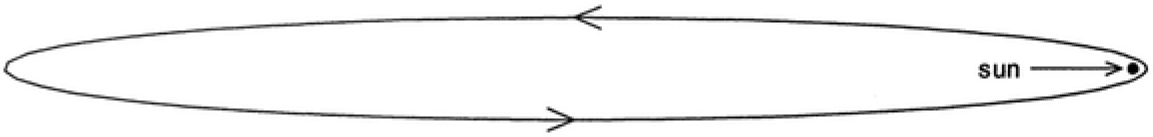


Figure S8.20

8.21 ⋆⋆⋆ **(a)** If you haven't already done so, do Problem 8.20.
(b) Use Kepler's third law (8.55) to find the period of this orbit in terms of $r_{\max}$ (and G and M_s).
(c) Now consider the extreme case that the comet is released from *rest* at a distance $r_{\max}$ from the sun. (In this case ℓ is actually zero.) Use the technique described in connection with (4.58) to find how long the comet takes to reach the sun. (Take the sun's radius to be zero.)
(d) Assuming the comet can somehow pass freely through the sun, describe its overall motion and find its period.
(e) Compare your answers in parts (b) and (d).

(a) See solution to Problem 8.20 above.
(b) For the narrow elliptical orbit, with $\ell \to 0$ but $\ell \neq 0$, we've seen that $a = r_{\max}/2$. Thus, by Kepler's third law, (8.55), $\tau_{(\ell\to 0)} = \pi(r_{\max})^{3/2}/\sqrt{2GM_s}$.
(c) With $\ell = 0$, the problem is one-dimensional. When the comet is a distance r from the sun, its energy is $\frac{1}{2}mv^2 - \gamma/r = E = -\gamma/r_{\max}$, where, as usual $\gamma = GM_s m$. Therefore, as it falls toward the sun (segment 1 in the figure) its radial velocity is $v_r = -\sqrt{2GM_s}\sqrt{1/r - 1/r_{\max}}$. (The minus sign is because it is moving inward.) The time to fall from $r_{\max}$ to the center of the sun is

$$t = \int_{r_{\max}}^{0} \frac{dr}{v_r} = \frac{1}{\sqrt{2GM_s}} \int_0^{r_{\max}} \frac{dr}{\sqrt{1/r - 1/r_{\max}}} = \frac{\pi (r_{\max})^{3/2}}{2\sqrt{2GM_s}} .$$

(To do the integral, make the sustitution $r/r_{\max} = u$.)

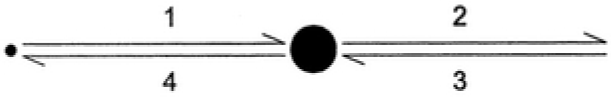


Figure S8.21

(d) When the comet passes the center of the sun it heads straight out the other side (segment 2 in the picture) and moves out to the same distance $r_{\max}$. Next it falls back to the center (segment 3), and finally it passes through and moves back to its starting point (segment 4). The time to do all this is 4 times the time t of part (c), so the period is $\tau_{(\ell=0)} = 2\pi (r_{\max})^{3/2}/\sqrt{2GM_s}$.
(e) The period $\tau_{(\ell=0)}$ is twice as long as $\tau_{(\ell\to 0)}$. This is because if the comet has ℓ small but nonzero, then when it reaches the sun it swings around the sun and effectively jumps from segment 1 to segment 4. If it has ℓ exactly zero, it doesn't know which side to swing around and it effectively doubles its journey by going straight through and out on segments 2 and 3.

8.23 ⋆⋆⋆ A particle of mass m moves with angular momentum ℓ in the field of a fixed force center with

$$F(r) = -\frac{k}{r^2} + \frac{\lambda}{r^3}$$

where k and λ are positive. **(a)** Write down the transformed radial equation (8.41) and prove that the orbit has the form

$$r(\phi) = \frac{c}{1 + \epsilon \cos(\beta\phi)}$$

where c, β, and ϵ are positive constants. **(b)** Find c and β in terms of the given parameters, and describe the orbit for the case that $0 < \epsilon < 1$. **(c)** For what values of β is the orbit closed? What happens to your results as $\lambda \to 0$?

(a) & (b) Since $F = -ku^2 + \lambda u^3$, the transformed radial equation is

$$u''(\phi) = -u - \frac{mF}{\ell^2 u^2} = -u + \frac{mk}{\ell^2} - \frac{m\lambda}{\ell^2} u = -\beta^2 u + \frac{mk}{\ell^2} = -\beta^2 (u - K),$$

where for the third equality I have defined $\beta = \sqrt{1 + m\lambda/\ell^2}$ and, for the fourth, $K = mk/(\ell\beta)^2$. The general solution is $u = K + A\cos(\beta\phi - \delta)$, but by redefining the direction $\phi = 0$ we can make $\delta = 0$. If we define $\epsilon = A/K$, then $u = K(1 + \epsilon\cos\beta\phi)$ and we can write $r = 1/u = c/(1 + \epsilon\cos\beta\phi)$, where $c = 1/K = \ell^2\beta^2/mk$.

The denominator of r is periodic in ϕ with period $2\pi/\beta < 2\pi$; if $0 < \epsilon < 1$, the denominator never vanishes, so r oscillates between minimum and maximum values, starting out at a minimum when $\phi = 0$. Thus, the particle "tries" to follow an ellipse, but r returns to its minimum before ϕ has made a complete revolution.
(c) If β is a rational number, $\beta = p/q$ (where p and q are integers), then the orbit will close after q revolutions. If β is irrational, the orbit will never close. If $\lambda \to 0$, then $\beta \to 1$ and the orbit becomes a Kepler ellipse.

8.25 ⋆⋆⋆ [Computer] Consider a particle with mass m and angular momentum ℓ in the field of a central force $F = -k/r^{5/2}$. To simplify your equations, choose units for which $m = \ell = k = 1$. **(a)** Find the value r_o of r at which U_{eff} is minimum and make a plot of $U_{\text{eff}}(r)$ for $0 < r \leq 5r_o$. (Choose your scale so that your plot shows the interesting part of the curve.) **(b)** Assuming now that the particle has energy $E = -0.1$, find an accurate value of r_{min}, the particle's distance of closest approach to the force center. (This will require the use of a computer program to solve the relevant equation numerically.) **(c)** Assuming that the particle is at $r = r_{\text{min}}$ when $\phi = 0$, use a computer program (such as "NDSolve" in Mathematica) to solve the transformed radial equation (8.41) and find the orbit in the form $r = r(\phi)$ for $0 \leq \phi \leq 7\pi$. Plot the orbit. Does it appear to be closed?

(a) The effective potential energy $U_{\text{eff}}(r)$ is

$$U_{\text{eff}} = U_{\text{cf}} + U = \frac{\ell^2}{2mr^2} - \frac{2k}{3r^{3/2}} = \frac{1}{2r^2} - \frac{2}{3r^{3/2}}$$

where the last expression results from the choice of units with $m = \ell = k = 1$. This function is plotted in the left picture below. The derivative is $U'_{\text{eff}}(r) = -r^{-3} + r^{-5/2}$, and this vanishes at $r = 1$. That is, the effective PE is minimum at $r_o = 1$ as indicated.

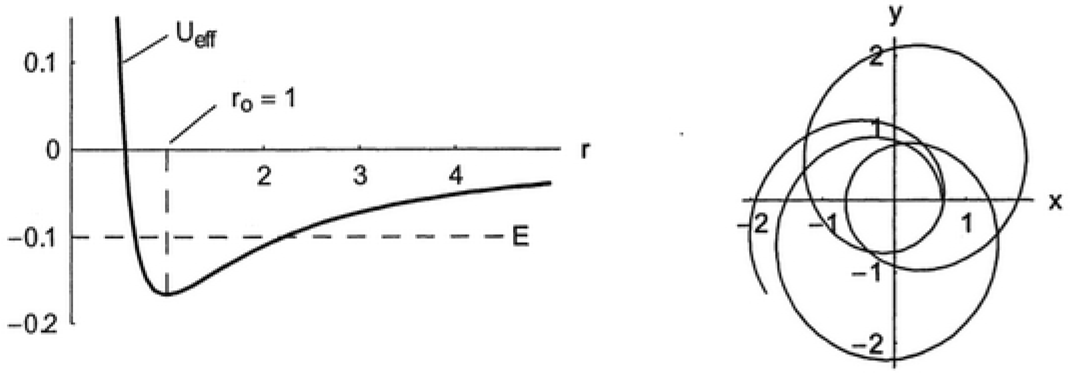


Figure S8.25

(b) You can see from the left picture that if $E = -0.1$, the inner turning point is at about $r_{\text{min}} = 0.7$. If we use this as a starting value for any equation-solving program (such as Mathematica's FindRoot), we find that this root of the equation $U_{\text{eff}}(r) = -0.1$ is actually $r_{\text{min}} = 0.6671$.

(c) The transformed radial equation (8.41) reads $u'' = -u + u^{1/2}$, with the initial conditions $u(0) = 1/r_{\text{min}}$ and $u'(0) = 0$. This has to be solved numerically and produces the orbit shown on the right. Obviously this orbit has not closed after 3.5 revolutions, and we can see that it won't close for a very long time: If you look closely, you can see that r has returned to r_{min} at about $\phi = 2.9\pi$ (actually 2.8991π to 5 significant figures). Clearly, therefore, it cannot return to r_{min} at an integer multiple of 2π for at least 20,000 revolutions. (In fact, it never does, but this is harder to prove.)

8.27 ⋆⋆⋆ At time t_o a comet is observed at radius r_o traveling with speed v_o at an acute angle α to the line from the comet to the sun. Put the sun at the origin O, with the comet on the x axis (at t_o) and its orbit in the xy plane, and then show how you could calculate the parameters of the orbital equation in the form $r = c/[1 + \epsilon\cos(\phi - \delta)]$. Do so for the case that $r_o = 1.0 \times 10^{11}$ m, $v_o = 45$ km/s, and $\alpha = 50$ degrees. [The sun's mass is about 2.0×10^{30} kg, and $G = 6.7 \times 10^{-11}$ N·m^2/s^2.]

We are given the initial speed and radius, v_o and r_o, and the angle α between $\mathbf{v}_o$ and the line from the comet to the sun. (And we know G and the sun's mass M.) If we denote the comet's unknown mass by m, then, in terms of m and the givens, we can calculate the following:

$$T_o = \tfrac{1}{2}mv_o^2 \quad \text{and} \quad U_o = -GMm/r_o$$

and hence

$$E = T_o + U_o, \quad \ell = mv_o r_o \sin\alpha, \quad \text{and} \quad c = \ell^2/(GMm^2).$$

From Eq. (8.58),

$$\epsilon = \sqrt{1 + 2E\ell^2/(G^2M^2m^3)}$$

and since $r = c/[1 + \epsilon\cos(\phi - \delta)]$

$$\delta = \arccos[(c/r_o - 1)/\epsilon].$$

If you look carefully, you'll see that the unknown mass m cancels from the three parameters c, ϵ, and δ. (So, in practice, we may as well set $m = 1$.) Thus we can find the numerical values of these parameters, and we've got the orbit. With the given numbers, we get

$$c = 8.87 \times 10^{10}\ \text{m}, \quad \epsilon = 0.753, \quad \text{and} \quad \delta = 1.72.$$

The figure shows distances in units of 10^{11} m.

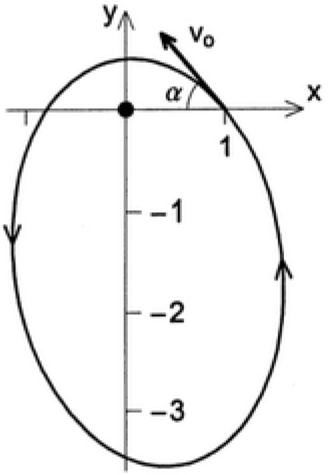


Figure S8.27

SECTION 8.7 The Unbounded Kepler Orbits

8.29 ⋆⋆ What would become of the earth's orbit (which you may consider to be a circle) if half of the sun's mass were suddenly to disappear? Would the earth remain bound to the sun? [*Hints:* Consider what happens to the earth's KE and PE at the moment of the great disappearance. The virial theorem for the circular orbit (Problem 4.41) helps with this one.] Treat the sun (or what remains of it) as fixed.

When the mass of the sun is suddenly halved, the earth's PE is immediately halved, $U = \frac{1}{2}U_o$. On the other hand, the KE is unchanged, $T = T_o$. Therefore, the total energy becomes $E = T + U = T_o + \frac{1}{2}U_o = 0$, because $T_o = -\frac{1}{2}U_o$ in a circular orbit (virial theorem). Since the final orbit has $E = 0$, it is a parabola, and the earth would eventually escape from the sun.

8.31 ★★★ Consider the motion of two particles subject to a *repulsive* inverse-square force (for example, two positive charges). Show that this system has no states with $E < 0$ (as measured in the CM frame), and that in all states with $E > 0$, the relative motion follows a hyperbola. Sketch a typical orbit. [*Hint:* You can follow closely the analysis of Sections 8.6 and 8.7 except that you must reverse the force; probably the simplest way to do this is to change the sign of γ in (8.44) and all subsequent equations (so that $F(r) = +\gamma/r^2$) and then keep γ itself positive. Assume $\ell \neq 0$.]

If we write $F_r = \gamma/r^2$ (with γ positive), then $E = T + U = T + \gamma/r$, which is never negative. The transformed radial equation (8.41) reads $u'' = -u - \gamma\mu/\ell^2$. [Compare (8.45).] This has the solution

$$r = \frac{1}{u} = \frac{c}{\epsilon\cos\phi - 1}, \tag{S8.4}$$

where $c = \ell^2/\gamma\mu > 0$ and ϵ is a positive constant. [In general the cosine has the form $\cos(\phi - \delta)$, but by redefining the zero of ϕ we can make $\delta = 0$.] Before showing that this is a hyperbola, we need to note that, as you can easily check, ϵ is related to E as in (8.58), $E = (\gamma^2\mu/2\ell^2)(\epsilon^2 - 1)$. Since $E \geq 0$, it follows that $\epsilon \geq 1$. If $\epsilon = 1$, it is easy to check that (S8.4) defines a parabola. If $\epsilon > 1$, then multiply (S8.4) by $\epsilon\cos\phi - 1$ to give $\epsilon x - r = c$, which becomes (after some algebra)

$$\frac{(x-d)^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $a = c/(\epsilon^2 - 1)$, $b = a\sqrt{\epsilon^2 - 1}$, and $d = a\epsilon$. This is the equation of a hyperbola, as in the sketch, which shows the path of the relative position $\mathbf{r} = (x, y)$, that is, the orbit of particle 1 as seen from particle 2.

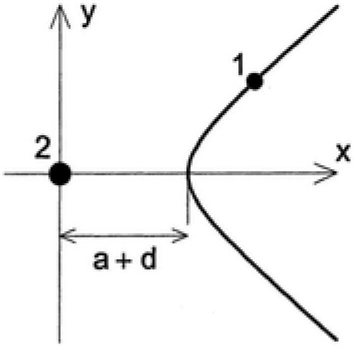


Figure S8.31

SECTION 8.8 Changes of Orbit

8.33 ★★ Figure 8.13 shows a space vehicle boosting from a circular orbit 1 at P to a transfer orbit 2 and then from the transfer orbit at P' to the final circular orbit 3. Example 8.6 derived in detail the thrust factor required for the boost at P. Show similarly that the thrust factor required at P' is $\lambda' = \sqrt{(R_1 + R_3)/2R_1}$. [Your argument should parallel closely that of Example 8.6, but you must account for the fact that P' is the apogee (not perigee) of the transfer orbit. For example, the plus signs in (8.67) should be minus signs here.]

The boost here occurs at P' which is the apogee, so the continuity of the orbit implies that $c_2/(1 - \epsilon_2) = c_3/(1 - \epsilon_3) = c_3$, since $\epsilon_3 = 0$. If the boost factor at P' is λ', then $\ell_3 = \lambda'\ell_2$, which implies that $c_3 = \lambda'^2 c_2$. Combining these two results, we get $\lambda'^2 = 1/(1 - \epsilon_2)$. But we already know that $\epsilon_2 = \lambda^2 - 1$, and that $\lambda^2 = 2R_3/(R_1 + R_3)$. Combining these, we conclude that $\lambda'^2 = 1/(2 - \lambda^2) = (R_1 + R_3)/2R_1$, as claimed.

8.35 ⋆⋆⋆ A spacecraft in a circular orbit wishes to transfer to another circular orbit of quarter the radius by means of a tangential thrust to move into an elliptical orbit and a second tangential thrust at the opposite end of the ellipse to move into the desired circular orbit. (The picture looks like Figure 8.13 but run backwards.) Find the thrust factors required and show that the speed in the final orbit is two times *greater* than the initial speed.

The simplest way to do this problem is to imagine Example 8.6 run backwards and to invert the thrust factors found there. The first thrust comes at the point P' of Figure 8.13, and its thrust factor is [compare (8.73)] $\lambda' = \sqrt{2R_1/(R_1 + R_3)} = \sqrt{2/5} = 0.63$, where R_3 denotes the larger radius and R_1 the smaller, as in Figure 8.13. Similarly, the second thrust comes at P and its thrust factor is [compare (8.72)] $\lambda = \sqrt{(R_1 + R_3)/2R_3} = \sqrt{5/8} = 0.79$. The speed changes during the transfer orbit, and, by conservation of angular momentum, $v(\text{at } P) = v(\text{at } P')R_3/R_1$. Therefore

$$v_{\text{fin}} = \lambda \cdot \frac{R_3}{R_1} \cdot \lambda' \cdot v_{\text{in}} = \sqrt{\frac{R_1 + R_3}{2R_3}} \cdot \frac{R_3}{R_1} \cdot \sqrt{\frac{2R_1}{R_1 + R_3}} \cdot v_{\text{in}} = \sqrt{\frac{R_3}{R_1}} \cdot v_{\text{in}} = 2v_{\text{in}}$$

CHAPTER 9

Mechanics in Noninertial Frames

PROBLEMS AND SOLUTIONS

SECTION 9.1 Acceleration without Rotation

9.1 ⋆ Be sure you understand why a pendulum in equilibrium hanging in a car that is accelerating forward tilts backward, and then consider the following: A helium balloon is anchored by a massless string to the floor of a car that is accelerating forward with acceleration A. Explain clearly why the balloon tends to tilt *forward* and find its angle of tilt in equilibrium. [*Hint:* Helium balloons float because of the buoyant Archimedean force, which results from a pressure gradient in the air. What is the relation between the directions of the gravitational field and the buoyant force?]

The bob of a pendulum at rest in an inertial frame is subject to two forces, the tension $\mathbf{T}$ in the string and the force of gravity $m\mathbf{g}$. Therefore, in equilibrium $\mathbf{T} + m\mathbf{g} = 0$ or $\mathbf{T} = -m\mathbf{g}$; that is, the tension force must be vertically up. Since the direction of the tension force in a string is always the direction of the string, the string must point vertically up from the bob. That is, the pendulum must hang vertically down.

We know that a pendulum in a car that is accelerating forward can be treated in the same way, provided we replace $\mathbf{g}$ with $\mathbf{g}_{\text{eff}} = \mathbf{g} - \mathbf{A}$, which points down and backward. By the same argument, this means the string must point up and forward from the bob, and the pendulum as a whole tilts backward.

With a pendulum it is legitimate to ignore the buoyant force $\mathbf{F}_{\text{b}}$ of the air on the bob. With a helium balloon it is certainly not, since $\mathbf{F}_{\text{b}}$ is actually greater than the weight. Thus the condition for equilibrium of a helium balloon in a non-accelerating car is

$$\mathbf{T} = -(m\mathbf{g} + \mathbf{F}_{\text{b}}). \tag{S9.1}$$

The direction of $\mathbf{F}_{\text{b}}$ is opposite to the pressure gradient, which is in the direction of $\mathbf{g}$. ($\mathbf{F}_{\text{b}}$ pushes toward lower pressures.) Thus $\mathbf{F}_{\text{b}}$ is vertically upward and greater in magnitude than $m\mathbf{g}$. Therefore the right side of Eq. (S9.1) is downward, and the tension must be pulling down on the balloon. That is, the balloon floats vertically upward.

If the car is accelerating forward we have only to replace $\mathbf{g}$ in (S9.1) with $\mathbf{g}_{\text{eff}}$, which points down and back. Now the buoyant force is opposite to $\mathbf{g}_{\text{eff}}$, so $\mathbf{F}_{\text{b}}$ points up and forward. Since $\mathbf{F}_{\text{b}}$ is greater in magnitude than $m\mathbf{g}$, Eq. (S9.1) implies that $\mathbf{T}$ must point down and back, which means the balloon has to be floating up and forward. The direction of the string is that of $\mathbf{g}_{\text{eff}}$, so the angle of tilt is $\arctan(A/g)$.

SECTION 9.2 The Tides

9.3 ⋆⋆ **(a)** Consider the tidal force (9.12) on a mass m at the position P of Figure 9.4. Write d as $(d_{\text{o}} - R_{\text{e}}) = d_{\text{o}}(1 - R_{\text{e}}/d_{\text{o}})$ and use the binomial approximation $(1 - \epsilon)^{-2} \approx 1 + 2\epsilon$ to show that

$\mathbf{F}_{\text{tid}} \approx -(2GM_{\text{m}}mR_{\text{e}}/d_{\text{o}}^3)\,\hat{\mathbf{x}}$. Confirm the direction of the force shown in Figure 9.4 and make a numerical comparison of the tidal force with the gravitational force $m\mathbf{g}$ of the earth. **(b)** Do the corresponding calculations for the force at the point R. Compare this force with that of part (a) (magnitude and direction).

(a) Referring to Fig.9.3, we see that, as suggested, the distance d is $d = d_{\text{o}} - R_{\text{e}} = d_{\text{o}}(1 - R_{\text{e}}/d_{\text{o}})$ at the point P, and clearly $\hat{\mathbf{d}} = \hat{\mathbf{d}}_{\text{o}} = \hat{\mathbf{x}}$. Therefore, by Eq. (9.12)

$$\mathbf{F}_{\text{tid}} = -\frac{GM_{\text{m}}m}{d_{\text{o}}^2}\left(\frac{1}{(1 - R_{\text{e}}/d_{\text{o}})^2} - 1\right)\hat{\mathbf{x}}. \tag{S9.2}$$

Obviously this has the direction of $-\hat{\mathbf{x}}$ as claimed. If we use the binomial approximation to write $(1 - R_{\text{e}}/d_{\text{o}})^{-2} \approx 1 + 2R_{\text{e}}/d_{\text{o}}$, this gives

$$\mathbf{F}_{\text{tid}} = -2\frac{GM_{\text{m}}mR_{\text{e}}}{d_{\text{o}}^3}\hat{\mathbf{x}}. \tag{S9.3}$$

Since the gravitational force of the earth has magnitude $mg = GM_{\text{e}}m/R_{\text{e}}^2$, their ratio is

$$\frac{F_{\text{tid}}}{mg} = 2\frac{M_{\text{m}}}{M_{\text{e}}}\left(\frac{R_{\text{e}}}{d_{\text{o}}}\right)^3 = 2 \times \frac{7.35 \times 10^{22}}{5.98 \times 10^{24}} \times \left(\frac{6.37 \times 10^6}{3.84 \times 10^8}\right)^3 = 1.1 \times 10^{-7}.$$

(b) At the point R of Fig.9.4, $d = d_{\text{o}} + R_{\text{e}}$. Therefore $\mathbf{F}_{\text{tid}}$ is given by Eq. (S9.2) except that the denominator has a plus sign instead of a minus. Clearly $\mathbf{F}_{\text{tid}}$ is now in the direction of $+\hat{\mathbf{x}}$. If we make the binomial approximation, then we get the same answer as in (S9.3) except that the minus sign has become a plus. That is, the tidal force at R is equal and opposite to that at P.

9.5 ⋆⋆ Review the derivation of the tidal potential energy (9.16) of a drop of water at the point Q in Figure 9.5 and then give in detail the derivation of (9.17) for the tidal PE at the point P.

At the point P, $d = d_{\text{o}} - R_{\text{e}}$ and $x = -R_{\text{e}}$. Therefore

$$\begin{aligned} U_{\text{tid}} = -GM_{\text{m}}m\left(\frac{1}{d} + \frac{x}{d_{\text{o}}^2}\right) &= -GM_{\text{m}}m\left(\frac{1}{d_{\text{o}} - R_{\text{e}}} - \frac{R_{\text{e}}}{d_{\text{o}}^2}\right) \\ &= -\frac{GM_{\text{m}}m}{d_{\text{o}}}\left(\frac{1}{1 - R_{\text{e}}/d_{\text{o}}} - \frac{R_{\text{e}}}{d_{\text{o}}}\right) \approx -\frac{GM_{\text{m}}m}{d_{\text{o}}}\left(1 + \frac{R_{\text{e}}^2}{d_{\text{o}}^2}\right) \end{aligned}$$

where in the last equality I used the binomial series to write $(1 - \epsilon)^{-1} \approx 1 + \epsilon + \epsilon^2$.

SECTION 9.4 Time Derivatives in a Rotating Frame

9.7 ⋆ **(a)** Explain the relation (9.30) between the derivatives of a vector $\mathbf{Q}$ in two frames $\mathcal{S}_{\text{o}}$ and $\mathcal{S}$ for the special case that $\mathbf{Q}$ is fixed in the frame $\mathcal{S}$. **(b)** Do the same for a vector $\mathbf{Q}$ that is fixed in the frame $\mathcal{S}_{\text{o}}$ and compare with your answer to part (a).

(a) If the vector $\mathbf{Q}$ is fixed in the frame $\mathcal{S}$, then $(d\mathbf{Q}/dt)_{\mathcal{S}} = 0$. Thus the relation (9.30)

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{\mathcal{S}_{\text{o}}} = \left(\frac{d\mathbf{Q}}{dt}\right)_{\mathcal{S}} + \boldsymbol{\Omega} \times \mathbf{Q}$$

reduces to

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{\mathcal{S}_o} = \boldsymbol{\Omega} \times \mathbf{Q}.$$

This is just the "useful relation" (9.22) for the rate of change, as seen in $\mathcal{S}_o$ of a vector $\mathbf{Q}$ fixed in a body rotating with angular velocity $\boldsymbol{\Omega}$.
(b) If $\mathbf{Q}$ is fixed in $\mathcal{S}_o$, then $(d\mathbf{Q}/dt)_{\mathcal{S}_o} = 0$, and the relation (9.30) reduces to

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{\mathcal{S}} = -\boldsymbol{\Omega} \times \mathbf{Q}.$$

Again this is just the "useful relation" (9.22), except that $\mathbf{Q}$ is fixed in $\mathcal{S}_o$ and the angular velocity of $\mathcal{S}_o$ relative to $\mathcal{S}$ is $-\boldsymbol{\Omega}$.

SECTION 9.5 Newton's Second Law in a Rotating Frame

9.9 ⋆ A bullet of mass m is fired with muzzle speed v_o horizontally and due north from a position at colatitude θ. Find the direction and magnitude of the Coriolis force in terms of m, v_o, θ, and the earth's angular velocity Ω. How does the Coriolis force compare with the bullet's weight if $v_o = 1000$ m/s and $\theta = 40$ deg?

$\mathbf{F}_{\text{cor}} = 2m\mathbf{v} \times \boldsymbol{\Omega} = 2mv_o\Omega\cos\theta$ due east, and

$$\frac{F_{\text{cor}}}{mg} = \frac{2v_o\Omega\cos\theta}{g} = \frac{2 \times (1000\text{ m/s}) \times (7.3 \times 10^{-5}\text{ rad/s}) \times (\cos 40°)}{9.8\text{ m/s}^2} = 0.0114.$$

9.11 ⋆⋆⋆ In this problem you will prove the equation of motion (9.34) for a rotating frame using the Lagrangian approach. As usual, the Lagrangian method is in many ways easier than the Newtonian (except that it calls for some slightly tricky vector gymnastics), but is perhaps less insightful. Let $\mathcal{S}$ be a noninertial frame rotating with constant angular velocity $\boldsymbol{\Omega}$ relative to the inertial frame $\mathcal{S}_o$. Let both frames have the same origin, $O = O'$. **(a)** Find the Lagrangian $\mathcal{L} = T - U$ in terms of the coordinates $\mathbf{r}$ and $\dot{\mathbf{r}}$ of $\mathcal{S}$. [Remember that you must first evaluate T in the inertial frame. In this connection, recall that $\mathbf{v}_o = \mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}$.] **(b)** Show that the three Lagrange equations reproduce (9.34) precisely.

(a) The KE evaluated in the inertial frame $\mathcal{S}_o$ is $T = \frac{1}{2}m\mathbf{v}_o^2 = \frac{1}{2}m(\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r})^2$, so the Lagrangian is $\mathcal{L} = \frac{1}{2}m(\dot{\mathbf{r}} + \boldsymbol{\Omega} \times \mathbf{r})^2 - U$.
(b) The derivatives of $\mathcal{L}$ are as follows:

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial x} &= m(\dot{\mathbf{r}} + \boldsymbol{\Omega} \times \mathbf{r}) \cdot \frac{\partial}{\partial x}(\boldsymbol{\Omega} \times \mathbf{r}) - \frac{\partial U}{\partial x} \\ &= m(\dot{\mathbf{r}} + \boldsymbol{\Omega} \times \mathbf{r}) \cdot (0, \Omega_z, -\Omega_y) - \frac{\partial U}{\partial x} \\ &= m[(\dot{\mathbf{r}} + \boldsymbol{\Omega} \times \mathbf{r}) \times \boldsymbol{\Omega}]_x - \frac{\partial U}{\partial x}.\end{aligned}$$

We can combine this with the two corresponding y and z equations, into a single vector equation

$$\frac{\partial\mathcal{L}}{\partial\mathbf{r}} = m\dot{\mathbf{r}} \times \boldsymbol{\Omega} + m(\boldsymbol{\Omega} \times \mathbf{r}) \times \boldsymbol{\Omega} + \mathbf{F}$$

Similarly

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}} = m(\ddot{\mathbf{r}} + \mathbf{\Omega} \times \dot{\mathbf{r}})$$

So the three Lagrange equations are $m\ddot{\mathbf{r}} = \mathbf{F} + 2m(\dot{\mathbf{r}} \times \mathbf{\Omega}) + m(\mathbf{\Omega} \times \mathbf{r}) \times \mathbf{\Omega}$, in agreement with Eq. (9.34).

SECTION 9.6 The Centrifugal Force

9.13 ⋆ Show that the angle α between a plumb line and the direction of the earth's center is given by $\sin\alpha = (R_e\Omega^2 \sin 2\theta)/(2g)$, where g is the observed free-fall acceleration and we assume the earth is perfectly spherically symmetric. Estimate the maximum and minimum values of the magnitude of α.

The radial and tangential components of $\mathbf{g}$ are given by (9.45) and (9.47) as

$$g_{\text{rad}} = g_o - \Omega^2 R \sin^2\theta \approx g_o \approx g \qquad \text{and} \qquad g_{\text{tang}} = \Omega^2 R \sin\theta\cos\theta = \tfrac{1}{2}\Omega^2 R \sin(2\theta)$$

where I use "radial" to mean the direction of the center of the earth. From Figure 9.11, we see that

$$\tan\alpha = \frac{g_{\text{tang}}}{g_{\text{rad}}} = \frac{\Omega^2 R}{2g}\sin(2\theta).$$

Thus $|\alpha|$ is minimum at the poles and the equator, where it is zero, and maximum at latitude 45°, where it is $\Omega^2 R/2g \approx 0.1°$ as in (9.48).

9.15 ⋆⋆ On a certain planet, which is perfectly spherically symmetric, the free-fall acceleration has magnitude $g = g_o$ at the North Pole and $g = \lambda g_o$ at the equator (with $0 \le \lambda \le 1$). Find $g(\theta)$, the free-fall acceleration at colatitude θ as a function of θ.

At the equator, we know that $g = g_o - \Omega^2 R$, but we are told this is λg_o. Therefore $\Omega^2 R = (1-\lambda)g_o$. Now, according to (9.44), at colatitude θ,

$$\mathbf{g}(\theta) = -g_o\hat{\mathbf{r}} + \Omega^2 R \sin\theta\,\hat{\boldsymbol{\rho}} = -g_o[\hat{\mathbf{r}} + (\lambda - 1)\sin\theta\,\hat{\boldsymbol{\rho}}] = -g_o[\cos\theta\,\hat{\mathbf{z}} + \lambda\sin\theta\,\hat{\boldsymbol{\rho}}]$$

since $\hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{z}} + \sin\theta\,\hat{\boldsymbol{\rho}}$. Therefore, $g(\theta) = g_o\sqrt{\cos^2\theta + \lambda^2\sin^2\theta}$.

SECTION 9.7 The Coriolis Force

9.17 ⋆ Consider the bead threaded on a circular hoop of Example 7.6 (page 260), working in a frame that rotates with the hoop. Find the equation of motion of the bead, and check that your result agrees with Equation (7.69). Using a free-body diagram, explain the result (7.71) for the equilibrium positions.

As seen in a frame rotating with the hoop, there are five forces on the bead. The first three, all of which act in the plane of the hoop, are the bead's weight $m\mathbf{g}$, the centrifugal force $\mathbf{F}_{\text{cf}} = m\omega^2 R\sin\theta\,\hat{\boldsymbol{\rho}}$, and the normal force $\mathbf{N}$ (actually the component of the normal force in the plane of the hoop). The other two are the Coriolis force $\mathbf{F}_{\text{cor}}$ and the component of the normal force normal to the hoop (neither of which is shown in the picture). Since these last two both act normal to the hoop, they cancel one another and need not concern us further. The bead can move only in the tangential direction, and its equation of motion is $m a_{\text{tang}} = F_{\text{tang}}$ or

$$m R\ddot{\theta} = F_{\text{cf}}\cos\theta - mg\sin\theta = (m\omega^2 R\sin\theta)\cos\theta - mg\sin\theta,$$

whence $\ddot{\theta} = (\omega^2\cos\theta - g/R)\sin\theta$, in agreement with Eq. (7.69).

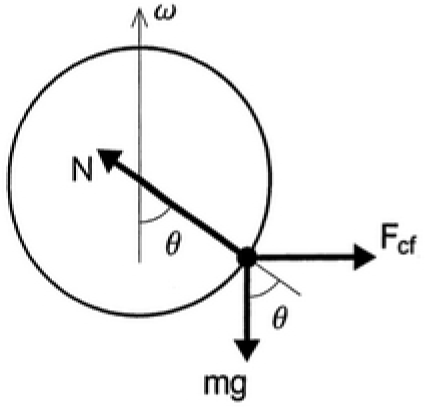


Figure S9.17

If the bead is in equilibrium, the tangential components of the centrifugal force and the weight must balance. That is, $F_{\text{cf}}\cos\theta = mg\sin\theta$, or $\cos\theta = g/(\omega^2 R)$, since $F_{\text{cf}} = m\omega^2 R\sin\theta$. This is the same as Eq. (7.71).

9.19 ⋆⋆ I am standing (wearing crampons) on a perfectly frictionless flat merry-go-round, which is rotating counterclockwise with angular velocity Ω about its vertical axis. **(a)** I am holding a puck at rest just above the floor (of the merry-go-round) and release it. Describe the puck's path as seen from above by an observer who is looking down from a nearby tower (fixed to the ground) and also as seen by me on the merry-go-round. In the latter case explain what I see in terms of the centrifugal and Coriolis forces. **(b)** Answer the same questions for a puck which is released from rest by a long-armed spectator who is standing on the ground leaning over the merry-go-round.

(a) As seen by a ground-based observer, the puck has initial velocity ΩR in the tangential direction. Since it is subject to zero net force, it travels in a straight line at constant speed (left picture). As seen from the merry-go-round, the puck is subject to the two inertial forces (centrifugal and Coriolis). It is initially at rest, so the Coriolis force is initially zero, and the puck is accelerated outward by the centrifugal force. As it speeds up, the Coriolis force becomes increasingly important and the puck curves to the right, spiralling outward.

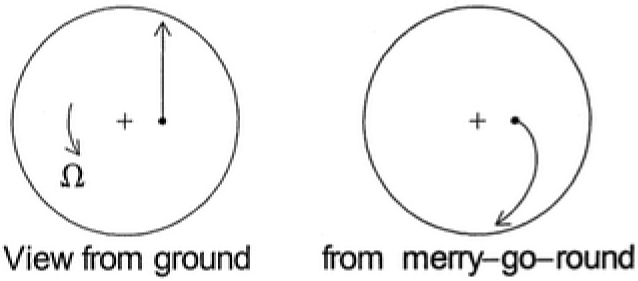


Figure S9.19a

(b) As seen from the ground, the puck is initially at rest. Since it is subject to zero net force, it remains at rest indefinitely. This means that, as seen from the merry-go-round, the puck describes a clockwise circle with angular velocity Ω and speed ΩR. This is quite a subtle result in the rotating frame. The centrifugal force is $m\Omega^2 R$ outward, and the Coriolis force is $2m\Omega^2 R$ inward; thus the net force is $m\Omega^2 R$ inward (as seen by observers on the merry-go-round), and this is just the required centripetal force to hold it in the circular orbit!

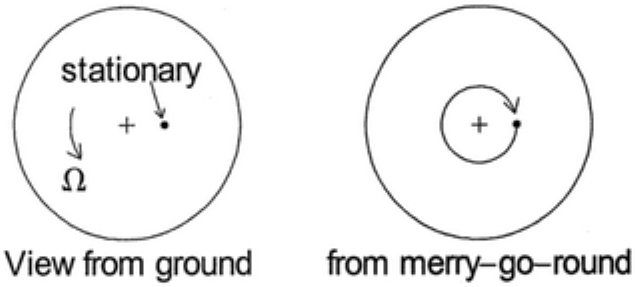


Figure S9.19b

9.21 ⋆⋆ When a frictionless puck slides on a rotating turntable, as in Problems 9.20 and 9.24, it can come instantaneously to rest. Sketch the shape of the path when this happens and explain. If you did Problem 9.24, comment on the relevance of this result to part (d) of that problem.

Suppose that the puck comes instantaneously to rest at a time t_o. As the puck comes to rest, the Coriolis force becomes less and less important, and the centrifugal force is what actually stops it. Since the centrifugal force is radially outward, the puck can only stop when it is headed inward, toward the axis of rotation O. Once it has stopped, the centrifugal force accelerates it radially outward again, and the orbit has the cusp shape shown. This is what happens in Problem 9.24(d).

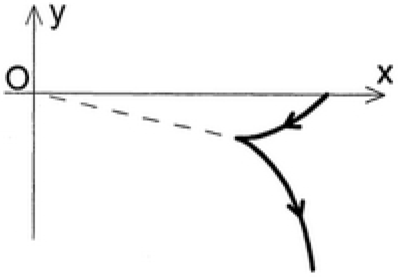


Figure S9.21

9.23 ⋆⋆ Here is an unusual way to solve the two-dimensional isotropic oscillator — the motion of a particle subject to a force $-k\mathbf{r}$. Show that by choosing a suitable rotating reference frame, you can arrange that the centrifugal force exactly cancels the force $-k\mathbf{r}$. Recalling the analogy between the Coriolis and magnetic forces, you should be able to write down the general solution for the motion as seen in the rotating frame. If you write your solution in the complex form of Section 2.7, then you can transform back to the nonrotating frame by multiplying by a suitable rotating complex number. Show that the general solution is an ellipse. [See Problem 8.11 for some guidance on this last part.]

Let $\mathcal{S}_o$ be the original inertial frame and choose the plane in which the motion occurs to be the $x_o y_o$ plane. Now consider a frame $\mathcal{S}$ with the same origin and z axis as $\mathcal{S}_o$ and with $\mathcal{S}$ rotating about the z_o axis with angular velocity Ω. For any object confined to the xy plane, the centrifugal force is $m\Omega^2\mathbf{r}$ and the equation of motion (in frame $\mathcal{S}$) is

$$m\ddot{\mathbf{r}} = -k\mathbf{r} + \mathbf{F}_{\rm cf} + \mathbf{F}_{\rm cor} = (-k + m\Omega^2)\mathbf{r} + 2m\dot{\mathbf{r}} \times \boldsymbol{\Omega}.$$

If we choose $\Omega = \sqrt{k/m}$ (the natural frequency of the mass on the spring), the centrifugal term exactly cancels the spring term and we're left with $m\ddot{\mathbf{r}} = 2m\dot{\mathbf{r}} \times \boldsymbol{\Omega}$, which is exactly the equation for a charge $q = 2m$ in a uniform magnetic field $\mathbf{B} = \boldsymbol{\Omega}$. In Section 2.7, we saw that the general solution has the form $x + iy = ce^{-i\omega t} + d$, where c and d are any two complex constants and $\omega = qB/m = 2\Omega$. [See the equation between (2.79) and (2.80).]

To translate this back to frame $\mathcal{S}_o$, notice that since $\mathcal{S}$ is rotating counterclockwise with angular velocity Ω, the complex coordinates satisfy $x_o + iy_o = (x + iy)e^{i\Omega t}$. Therefore

$$x_o + iy_o = (ce^{-2i\Omega t} + d)e^{i\Omega t} = ce^{-i\Omega t} + de^{i\Omega t} = a\cos\Omega t + b\sin\Omega t$$

where a and b are two other complex constants. Taking real and imaginary parts we conclude that

$$x_o = A\cos\Omega t + B\sin\Omega t \qquad \text{and} \qquad y_o = C\cos\Omega t + D\sin\Omega t$$

where A, B, C, D are four arbitrary real constants. This is the same solution as was found in the solution to Problem 8.11, and is an ellipse for the reasons given there.

SECTION 9.8 Free Fall and the Coriolis Force

9.25 ⋆ A high-speed train is traveling at a constant 150 m/s (about 300 mph) on a straight, horizontal track across the South Pole. Find the angle between a plumb line suspended from the ceiling inside the train and another inside a hut on the ground. In what direction is the plumb line on the train deflected?

At the south pole the centrifugal force is zero and the Coriolis force is horizontal and to the left of the train, with magnitude $F_{\text{cor}} = 2mv\Omega$. There are three forces on the mass at the end of the plumb line: $\mathbf{F}_{\text{cor}}$, the tension $\mathbf{T}$ in the line, and mass's weight $m\mathbf{g}$. (See the picture, which shows the plumb line as seen from the rear of the train. The train's velocity is into the page and the $\boldsymbol{\Omega}$ is vertically down.) The condition for equilibrium is

$$\mathbf{F}_{\text{cor}} + \mathbf{T} + m\mathbf{g} = 0.$$

For the plumb line in the stationary hut, $\mathbf{F}_{\text{cor}} = 0$, so $\mathbf{T} = -m\mathbf{g}$ and the line hangs vertically. On the train the angle with the vertical satisfies $\tan\alpha = F_{\text{cor}}/mg = 2v\Omega/g$, and the angle between the two lines is $\alpha = \arctan(2v\Omega/g) = 0.13°$ with the line on the train hanging toward the left.

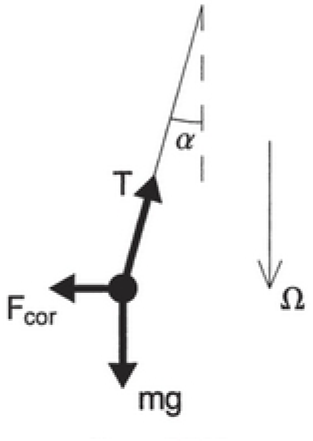
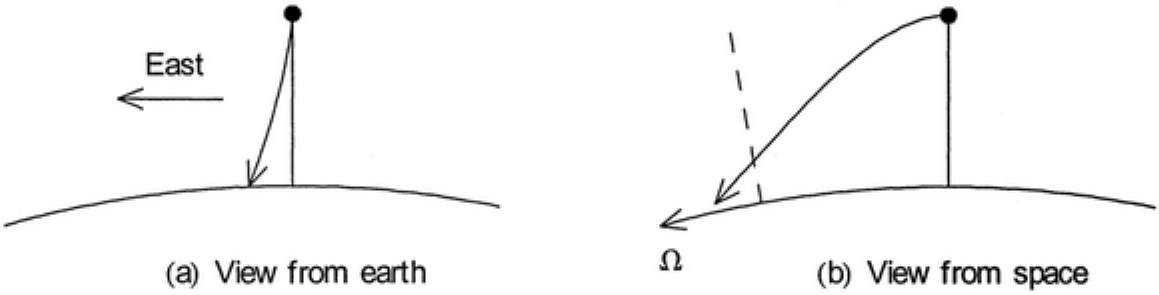


Figure S9.25

9.27 ⋆⋆ In Section 9.8, we discussed the path of an object that is dropped from a very tall stepladder above the equator. **(a)** Sketch this path as seen from a tower to the north of the drop and fixed to the earth. Explain why the object lands to the east of its point of release. **(b)** Sketch the same experiment as seen by an inertial observer floating in space to the north of the drop. Explain clearly (from this point of view) why the object lands to the east of its point of release. [*Hint:* The object's angular momentum about the earth's center is conserved. This means that the object's angular velocity $\dot{\phi}$ changes as it falls.]

(a) The dominant effect as the object falls is that it acquires an increasing velocity vertically downward. Therefore the Coriolis force $2m\mathbf{v} \times \mathbf{\Omega}$ (with $\mathbf{\Omega}$ pointing due north) points due east.The left picture shows the experiment as seen by someone standing north of the equator and looking south.



(a) View from earth (b) View from space

Figure S9.27

(b) As seen from space, the earth and the ladder on which the object initially rests are both rotating with angular velocity $\dot{\phi} = \Omega$ toward the east. When the object is released its angular momentum $\ell = mr^2\dot{\phi}$ must remain constant. Thus, as r gets smaller (slightly), $\dot{\phi}$ gets bigger, and the object moves slightly ahead of the ladder from which it was released. Therefore the object lands slightly to the east of the ladder's new position.

9.29 ⋆⋆ **(a)** A baseball is thrown vertically up with initial speed v_o from a point on the ground at colatitude θ. Use the solution (9.73) of Problem 9.26 to show that the ball will return to the ground a distance $(4\Omega v_o{}^3 \sin\theta)/(3g^2)$ to the west of its launch point. **(b)** Estimate the size of this effect on the equator if $v_o = 40$ m/s. **(c)** Sketch the ball's orbit as seen from the north (by an observer fixed to the earth). Compare with the orbit of a ball dropped from a point above the equator, and explain why the Coriolis effect moves the dropped ball to the east, but the thrown ball to the west.

(a) With $v_{xo} = v_{yo} = 0$ and $v_{zo} = v_o$, the trajectory given by (9.73) is

$$x = -\Omega \sin\theta(v_o - \tfrac{1}{3}gt)t^2, \qquad y = 0, \quad \text{and} \quad z = v_o t - \tfrac{1}{2}gt^2.$$

We see that the ball does not stray in the north-south (y) direction (at least to first order in Ω), but it does move in the east-west (x) direction. The time for the ball to return to the ground is found from the z equation to be $t_{gr} = 2v_o/g$. Substituting this value into the x equation, we find $x_{gr} = -(4\Omega v_o^3 \sin\theta)/(3g^2)$. Since this is negative, we conclude that the Coriolis effect causes the ball to land to the west of its point of departure.

(b) The westerly displacement is maximum at the equator, where $\sin\theta = 1$, and the displacement is

$$\frac{4\,\Omega\, v_o^3}{3\,g^2} = \frac{4 \times (7.3 \times 10^{-5}) \times (40)^3}{3 \times (9.8)^2} \approx 0.065 \text{ m}$$

or about 6 cm—not an effect that would be easy to detect.

(c) On the upward journey, the Coriolis force accelerates the ball to the west and on the downward journey to the east. Thus v_x starts from zero, increases (to the west) as the ball climbs, and decreases back to zero by the time the ball returns to the ground. Throughout the trip v_x is to the west, and the ball lands to the west of its starting position. The dropped ball starts with $v_x = 0$ at the top and its whole journey is downward, so that the Coriolis force accelerates it to the east throughout. Thus v_x is to the east at all times, and the ball lands to the east of its initial position. In the pictures, the Coriolis effect is much exaggerated.

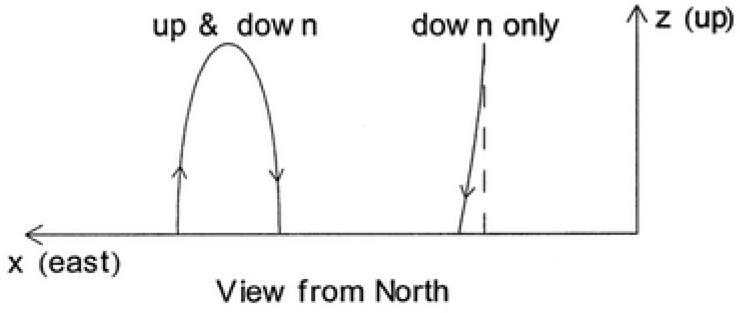


Figure S9.29

9.31 ★★★ The **Compton generator** is a beautiful demonstration of the Coriolis force due to the earth's rotation, invented by the American physicist A. H. Compton (1892–1962, best known as author of the Compton effect) while he was still an undergraduate. A narrow glass tube in the shape of a torus or ring (radius R of the ring $\gg$ radius of the tube) is filled with water, plus some dust particles to let one see any motion of the water. The ring and water are initially stationary and horizontal, but the ring is then spun through 180° about its east–west diameter. Explain why this should cause the water to move around the tube. Show that the speed of the water just after the 180° turn should be $2\Omega R\cos\theta$, where Ω is the earth's angular velocity, and θ is the colatitude of the experiment. What would this speed be if $R \approx 1$ m and $\theta = 40°$? Compton measured this speed with a microscope and got agreement within 3%.

The left picture shows the ring face-on, viewed from the north and above. The shaded segment of water has mass $dm = m\,d\alpha/2\pi$, where m is the total mass of water, and it experiences a Coriolis force $d\mathbf{F}$ to the east as it is swung toward the viewer. The right picture shows the ring side on, viewed from the west. The velocity of the shaded segment of water is $v = \dot\phi R \sin\alpha$ normal to the ring.

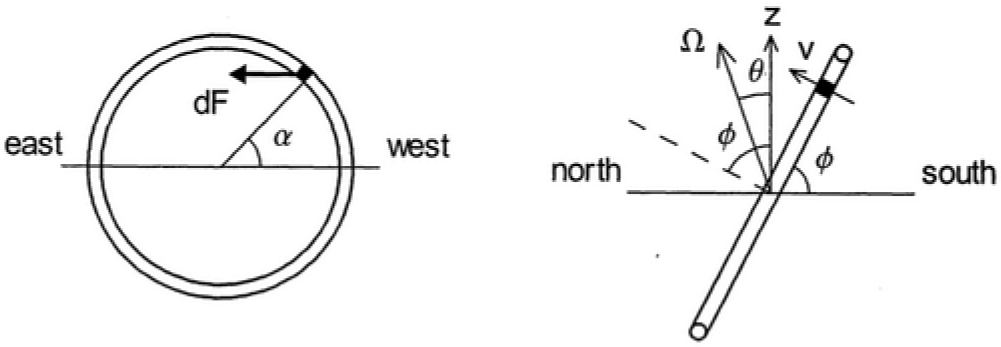


Figure S9.31

The force $d\mathbf{F}$ is

$$d\mathbf{F} = 2\,dm\,\mathbf{v}\times\mathbf{\Omega} = 2\,dm\,(\dot\phi R\sin\alpha)\Omega\sin(\phi-\theta)\text{ east.}$$

This produces a torque of magnitude

$$d\Gamma = |\mathbf{r} \times d\mathbf{F}| = R\,dF\,\sin\alpha = 2\,dm\,R^2\dot{\phi}\Omega\sin(\phi-\theta)\sin^2\alpha$$

tending to push the water counterclockwise as seen in the left picture. To find the total torque, we replace dm by $m\,d\alpha/2\pi$ and integrate from $\alpha = 0$ to 2π. This gives

$$\Gamma = \int d\Gamma = mR^2\dot{\phi}\Omega\sin(\phi-\theta).$$

The total angular momentum given to the water is

$$L = \int \Gamma dt = mR^2\Omega \int \sin(\phi-\theta)\dot{\phi}dt = mR^2\Omega \int_0^\pi \sin(\phi-\theta)d\phi = 2mR^2\Omega\cos\theta.$$

If V is the final speed of the water, then $L = mRV$, and equating these two expressions, we see that $V = 2R\Omega\cos\theta$. With the given numbers,

$$V = 2 \times (1\text{ m}) \times (7.3 \times 10^{-5}\text{ rad/s}) \times (\cos 40^\circ) = 0.11\text{ mm/s}.$$

SECTION 9.9 The Foucault Pendulum

9.33 ⋆⋆ The general solution for the small-amplitude motion of a Foucault pendulum is given by (9.66). If at $t = 0$ the pendulum is at rest with $x = A$ and $y = 0$, find the two coefficients C_1 and C_2, and show that because $\Omega \ll \omega_o$ they are well approximated as $C_1 = C_2 = A/2$, giving the solution (9.67).

According to Eq. (9.66), $\eta(t) = x + iy = e^{-i\Omega_z t}(C_1 e^{i\omega_o t} + C_2 e^{-i\omega_o t})$. The initial conditions are $\eta(0) = A$ and $\dot{\eta}(0) = 0$, which give us two equations for the two coefficients C_1 and C_2

$$C_1 + C_2 = A \qquad \text{and} \qquad -i\Omega_z(C_1 + C_2) + i\omega_o(C_1 - C_2) = 0.$$

These are easily solved to give

$$C_1 = \frac{A}{2}\left(1 + \frac{\Omega_z}{\omega_o}\right) \approx \frac{A}{2} \qquad \text{and} \qquad C_2 = \frac{A}{2}\left(1 - \frac{\Omega_z}{\omega_o}\right) \approx \frac{A}{2},$$

where the approximations follow because $\Omega_z/\omega_o \lesssim 10^{-4}$. (The period of a Foucault pendulum is of order 10 s, while that of the earth's spinning is 1 day $\sim 10^5$ s.)

CHAPTER 10

Rotational Motion of Rigid Bodies

PROBLEMS AND SOLUTIONS

SECTION 10.1 Properties of the Center of Mass

10.1 ⋆ The result (10.7), that $\sum m_\alpha \mathbf{r}'_\alpha = 0$, can be paraphrased to say that the position vector of the CM relative to the CM is zero, and, in this form, is nearly obvious. Nevertheless, to be sure you understand the result, prove it by solving (10.4) for $\mathbf{r}'_\alpha$ and substituting into the sum concerned.

From (10.4), $\mathbf{r}'_\alpha = \mathbf{r}_\alpha - \mathbf{R}$, so

$$\sum m_\alpha \mathbf{r}'_\alpha = \sum m_\alpha \mathbf{r}_\alpha - \sum m_\alpha \mathbf{R}.$$

Now, by the definition (10.1) of the CM position, the first sum on the right is just $M\mathbf{R}$, and, by factoring the $\mathbf{R}$ from the second sum, you can see that the second term is the same. Therefore, the two terms on the right cancel, and the sum on the left is zero.

10.3 ⋆ Five equal point masses are placed at the five corners of a square pyramid whose square base is centered on the origin in the xy plane, with side L, and whose apex is on the z axis at a height H above the origin. Find the CM of the five-mass system.

If we choose the x axis parallel to one of the base's edges, the positions of the five masses are

$$\tfrac{1}{2}(L, L, 0), \quad \tfrac{1}{2}(L, -L, 0), \quad \tfrac{1}{2}(-L, L, 0), \quad \tfrac{1}{2}(-L, -L, 0), \quad \text{and} \quad (0, 0, H)$$

Therefore

$$X = \frac{1}{M}\sum m x_\alpha = \frac{m}{M}\sum x_\alpha = 0, \qquad Y = \frac{1}{M}\sum m y_\alpha = \frac{m}{M}\sum y_\alpha = 0$$

and

$$Z = \frac{1}{M}\sum m z_\alpha = \frac{m}{M}\sum z_\alpha = \frac{1}{5}H.$$

10.4 ⋆⋆ The calculation of centers of mass or moments of inertia usually involves doing an integral, most often a volume integral, and such integrals are often best done in spherical polar coordinates (defined back in Figure 4.16). Prove that

$$\int dV f(\mathbf{r}) = \int r^2 dr \int \sin\theta \, d\theta \int d\phi \; f(r, \theta, \phi).$$

[Think about the small volume dV enclosed between r and $r + dr$, θ and $\theta + d\theta$, and ϕ and $\phi + d\phi$.] If the volume integral on the left runs over all space, what are the limits of the three integrals on the right?

10.5 ⋆⋆ A uniform solid hemisphere of radius R has its flat base in the xy plane, with its center at the origin. Use the result of Problem 10.4 to find the center of mass. [Comment: This and the next two problems are intended to reactivate your skills at finding centers of mass by integration. In all cases, you will need to use the integral form of the definition (10.1) of the CM. If the mass is distributed through a volume (as here), the integral will be a volume integral with $dm = \varrho\, dV$.]

The x and y coordinates are easy: For instance, $X = (1/M)\int \varrho x dV$. Since every point (x, y, z) in the hemisphere can be paired with a point $(-x, y, z)$ and the contributions from these two points exactly cancel, we conclude that $X = 0$. Similarly $Y = 0$. Finally

$$\begin{aligned} Z = \frac{1}{M}\int \varrho z dV &= \frac{\varrho}{M}\int_0^R r^2 dr \int_0^{\pi/2} \sin\theta d\theta \int_0^{2\pi} d\phi\, r\cos\theta \\ &= \frac{1}{V}\int_0^R r^3 dr \int_0^{\pi/2} \sin\theta\cos\theta d\theta \int_0^{2\pi} d\phi \;=\; \frac{3}{8}R \end{aligned}$$

where, in passing to the final result, I used the fact that $V = \frac{2}{3}\pi R^3$.

10.7 ⋆⋆ A "rounded cone" is made by cutting out of a uniform sphere of radius R the volume with $\theta \le \theta_o$, where θ is the usual angle measured from the polar axis and θ_o is a constant between 0 and π. **(a)** Describe this cone and use the result of Problem 10.4 to find its volume. **(b)** Find its CM and comment on your results for the cases that $\theta_o = \pi$ and $\theta_o \to 0$.

(a) The "rounded cone" is like an icecream cone that has been licked down until the top surface is nearly flat, but is actually part of a sphere centered on the bottom (apex) of the cone. Its volume is

$$V = \int dV = \int_0^R r^2 dr \int_0^{\theta_o} \sin\theta d\theta \int_0^{2\pi} d\phi = \tfrac{2}{3}\pi R^3(1 - \cos\theta_o)$$

If $\theta_o = \pi$, the "cone" is a sphere and our answer becomes the familiar $V = \frac{4}{3}\pi R^3$. When $\theta_o \to 0$ the rounded top of the cone becomes flat and, with $\cos\theta_o \approx 1 - \theta_o^2/2$, our answer becomes $V = \frac{1}{3}\pi R^3 \theta_o^2 = \frac{1}{3}\pi r^2 R$ where $r = R\theta_o$ is the radius of the top of the cone; this is the correct expression for the volume of an ordinary cone of radius r and height R.
(b) As in Problems 10.5 and 10.6, $X = Y = 0$ and

$$\begin{aligned} Z = \frac{1}{M}\int \varrho z dV &= \frac{\varrho}{M}\int_0^R r^2 dr \int_0^{\theta_o} \sin\theta d\theta \int_0^{2\pi} d\phi\, r\cos\theta \\ &= \frac{1}{V}\int_0^R r^3 dr \int_0^{\theta_o} \sin\theta\cos\theta d\theta \int_0^{2\pi} d\phi \;=\; \frac{3R}{16}\,\frac{1 - \cos 2\theta_o}{1 - \cos\theta_o} \end{aligned}$$

If $\theta_o = \pi$ the "cone" is a sphere and our answer becomes $Z = 0$; that is, the CM is at the origin, as expected. As $\theta_o \to 0$, the answer becomes $Z \to \frac{3}{4}R$, which is the correct distance from the apex of an ordinary cone to its CM.

SECTION 10.2 Rotation about a Fixed Axis

10.9 ⋆ The moment of inertia of a continuous mass distribution with density ϱ is obtained by converting the sum of (10.25) into the volume integral $\int \rho^2\, dm = \int \rho^2 \varrho\, dV$. (Note the two forms of the Greek "rho": $\rho =$ distance from z axis, $\varrho =$ mass density.) Find the moment of inertia of a uniform circular cylinder of radius R and mass M for rotation about its axis. Explain why the products of inertia are zero.

Let us put the cylinder with its axis on the z axis and use cylindrical polar coordinates. The density is $\varrho = M/V$ where $V = \pi R^2 h$, and the element of volume is $dV = \rho\, d\rho\, d\phi\, dz$. Thus

$$I_{zz} = \int \varrho \rho^2 dV = \frac{M}{V}\int_0^R \rho^3 d\rho \int_0^{2\pi} d\phi \int_0^h dz = \frac{M}{\pi R^2 h}\cdot\frac{R^4}{4}\cdot 2\pi\cdot h = \tfrac{1}{2}MR^2.$$

The products of inertia I_{xz} and I_{yz} are zero because the cylinder is axially symmetric about the z axis.

10.11 ⋆⋆ **(a)** Use the result of Problem 10.4 to find the moment of inertia of a uniform solid sphere (mass M, radius R) for rotation about a diameter. **(b)** Do the same for a uniform hollow sphere whose inner and outer radii are a and b. [One slick way to do this is to think of the hollow sphere as a solid sphere of radius b from which you have removed a sphere of the same density but radius a.]

(a) In terms of spherical polar coordinates, the distance from the axis out to a point (r, θ, ϕ) is $\rho = r \sin\theta$. Therefore, for a sphere of radius R,

$$I = \varrho \int \rho^2 dV = \varrho \int_0^R r^4 dr \int_0^\pi \sin^3\theta\, d\theta \int_0^{2\pi} d\phi = \frac{8\pi}{15}\varrho R^5 = \frac{2}{5}MR^2$$

since $\varrho = M/(\frac{4}{3}\pi R^3)$.

(b) If $I(b, a)$ denotes the moment of inertia of a hollow sphere of outer and inner radii b and a, then $I(b, 0)$ is that of a solid sphere of radius b and obviously $I(b, 0) = I(b, a) + I(a, 0)$. Therefore

$$I(b, a) = I(b, 0) - I(a, 0) = \frac{8\pi}{15}\varrho(b^5 - a^5) = \frac{2}{5}M\frac{b^5 - a^5}{b^3 - a^3}$$

since $\varrho = M/V = M/\left[\frac{4}{3}\pi(b^3 - a^3)\right]$.

10.13 ⋆⋆ A thin rod (of width zero, but not necessarily uniform) is pivoted freely at one end about the horizontal z axis, being free to swing in the xy plane (x horizontal, y vertically down). Its mass is m, its CM is a distance a from the pivot, and its moment of inertia (about the z axis) is I. **(a)** Write down the equation of motion $\dot{L}_z = \Gamma_z$ and, assuming the motion is confined to small angles (measured from the downward vertical), find the period of this compound pendulum. ("Compound pendulum" is traditionally used to mean any pendulum whose mass is distributed — as contrasted with a "simple pendulum," whose mass is concentrated at a single point on a massless arm.) **(b)** What is the length of the "equivalent" simple pendulum, that is, the simple pendulum with the same period?

(a) The equation $\dot{L}_z = \Gamma_z$ implies that $I\ddot{\phi} = -mga\sin\phi \approx -mga\phi$. This implies SHM with angular frequency $\omega = \sqrt{mga/I}$ and hence period $\tau = 2\pi\sqrt{I/(mga)}$.

(b) The period of a simple pendulum is $\tau = 2\pi\sqrt{L/g}$ and these two periods are equal if $L = I/(ma)$.

10.15 ⋆⋆ **(a)** Write down the integral (as in Problem 10.9) for the moment of inertia of a uniform cube of side a and mass M, rotating about an edge, and show that it is equal to $\frac{2}{3}Ma^2$. **(b)** If I balance the cube on an edge in unstable equilibrium on a rough table, it will eventually topple and rotate until it hits the table. By considering the energy of the cube, find its angular velocity just before it hits the table. (Assume the edge does not slide on the table.)

(a) $I = \frac{2}{3}Ma^2$. [See Example 10.2, Eq. (10.47)].

(b) The initial energy of the cube is pure PE and, since the CM is a height $a/\sqrt{2}$ above the table, $E_{\text{in}} = Mga/\sqrt{2}$. The KE just before impact is $\frac{1}{2}I\omega^2$ and the PE is $Mga/2$. Therefore, by conservation of energy

$$Mg\frac{a}{\sqrt{2}} = \frac{1}{3}Ma^2\omega^2 + Mg\frac{a}{2} \qquad \text{whence} \qquad \omega = \sqrt{\frac{3g}{2a}(\sqrt{2}-1)}.$$

10.17 ⋆⋆ Write down the integral for the moment of inertia of a uniform ellipsoid (mass M) with surface $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$ for rotation about the z axis. One simple way to do the integral is to make a change of variables to $\xi = x/a$, $\eta = y/b$, and $\zeta = z/c$. Each of the two resulting integrals can be related to the corresponding integrals for a sphere (as in Problem 10.11). Do this. Check your answer for the case $a = b = c$.

The moment of inertia for rotation about the z axis is $I_{zz} = \varrho \int dx \int dy \int dz\,(x^2 + y^2)$, where the integral runs through the interior of the given ellipsoid. If we make the suggested change of variables, then $dx = a\,d\xi$ and so on, and the integral becomes

$$\begin{aligned} I_{zz} &= \varrho abc \int d\xi \int d\eta \int d\zeta\,(a^2\xi^2 + b^2\eta^2) \\ &= \varrho abc \left(a^2 \int d\xi \int d\eta \int d\zeta\,\xi^2 + b^2 \int d\xi \int d\eta \int d\zeta\,\eta^2\right) \end{aligned} \tag{S10.1}$$

where the boundary for these new integrals is the unit sphere $\xi^2 + \eta^2 + \zeta^2 = 1$. Each of the two integrals in the second line can be easily evaluated. For example, in $\int d\xi \int d\eta \int d\zeta\,\xi^2$ the integral $\int d\eta \int d\zeta$ gives the area of a disk of radius $\sqrt{1-\xi^2}$, so that

$$\int d\xi \int d\eta \int d\zeta\,\xi^2 = \pi \int_{-1}^{1} d\xi (1-\xi^2)\xi^2 = \frac{4\pi}{15}.$$

The second integral in (S10.1) is the same, and we conclude that

$$I_{zz} = \frac{4}{15}\varrho\pi abc(a^2 + b^2) = \frac{1}{5}M(a^2 + b^2)$$

where I got the second equality by substituting $\varrho = M/(\frac{4}{3}\pi abc)$. (If you don't remember that the volume of an ellipsoid is $\frac{4}{3}\pi abc$, you can find it by doing the integral $\int dV$ using the same changes of variables.) Notice that for a sphere of radius R this reduces to the familiar $I_{zz} = \frac{2}{5}MR^2$.

SECTION 10.3 Rotation about Any Axis; the Inertia Tensor

10.19 ⋆ Verify that the components of the vector $\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})$ are given correctly by Equation (10.35). Do this both by working with components and by using the so-called $BAC - CAB$ rule, that is $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$.

(a) Because

$$\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z) \qquad \text{and} \qquad \mathbf{r} = (x, y, z)$$

it follows that

$$\boldsymbol{\omega} \times \mathbf{r} = \big(\omega_y z - \omega_z y,\ \omega_z x - \omega_x z,\ \omega_x y - \omega_y x\big)$$

and hence

$$[\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})]_x = y(\omega_x y - \omega_y x) - z(\omega_z x - \omega_x z) = (y^2 + z^2)\omega_x - xy\omega_y - xz\omega_z. \quad \text{(S10.2)}$$

This is exactly the x component of Eq. (10.35), and, since the other two components work the same way, we've proved (10.35).

(b) By the $BAC - CAB$ rule, $\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) = \boldsymbol{\omega}(\mathbf{r} \cdot \mathbf{r}) - \mathbf{r}(\mathbf{r} \cdot \boldsymbol{\omega})$, so

$$[\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})]_x = \omega_x(x^2 + y^2 + z^2) - x(x\omega_x + y\omega_y + z\omega_z).$$

On the right side, the first and fourth terms cancel and the remaining four reproduce the right side of Eq. (S10.2). Once again the other two components work the same way, and we've proved (10.35).

10.21 ⋆⋆ The definition of the inertia tensor in Equations (10.37) and (10.38) has the rather ugly feature that the diagonal and off-diagonal elements are defined by completely different equations. Show that the two definitions can be combined into the single equation (which is slightly less messy in integral form)

$$I_{ij} = \int \varrho(r^2\delta_{ij} - r_i r_j)dV$$

where δ_{ij} is the **Kronecker delta symbol**

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases} \quad (10.116)$$

Let us start from the given expression and show that it's elements *are* equal to the usual definitions (10.37) and (10.38). First

$$I_{xx} = \int \varrho(r^2 - xx)dV = \int \varrho(y^2 + z^2)dV$$

which is exactly the definition (10.37) (in integral form). The two other diagonal elements work the same way. Meanwhile,

$$I_{xy} = \int \varrho(0 - xy)dV = -\int \varrho(xy)dV$$

which is exactly the definition (10.38). The other off-diagonal elements all work the same way, and we've shown that the proposed definition agrees with the original one.

10.23 ⋆⋆ Consider a rigid plane body or "lamina," such as a flat piece of sheet metal, rotating about a point O in the body. If we choose axes so that the lamina lies in the xy plane, which elements of the inertia tensor **I** are automatically zero? Prove that $I_{zz} = I_{xx} + I_{yy}$.

Since the whole body lies in the plane $z = 0$, the four products of inertia involving z are all zero, $I_{xz} = I_{yz} = I_{zx} = I_{zy} = 0$. For example,

$$I_{xz} = -\sum m_\alpha x_\alpha z_\alpha = 0.$$

For the same reason,

$$I_{xx} + I_{yy} = \sum m_\alpha ({y_\alpha}^2 + {z_\alpha}^2) + \sum m_\alpha ({z_\alpha}^2 + {x_\alpha}^2) = \sum m_\alpha ({x_\alpha}^2 + {y_\alpha}^2) = I_{zz}.$$

10.24 ⋆⋆ **(a)** If $\mathbf{I}^{\text{cm}}$ denotes the moment of inertia tensor of a rigid body (mass M) about its CM, and $\mathbf{I}$ the corresponding tensor about a point P displaced from the CM by $\boldsymbol{\Delta} = (\xi, \eta, \zeta)$, prove that

$$I_{xx} = I_{xx}^{\text{cm}} + M(\eta^2 + \zeta^2) \qquad \text{and} \qquad I_{yz} = I_{yz}^{\text{cm}} - M\eta\zeta, \tag{10.117}$$

and so forth. (These results, which generalize the parallel-axis theorem that you probably learned in introductory physics, mean that once you know the inertia tensor for rotation about the CM, calculating it for any other origin is trivially easy.) **(b)** Confirm that the results of Example 10.2 (page 381) fulfill the identities (10.117) [so that the calculations of part (a) of the example were actually unnecessary].

10.25 ⋆⋆ **(a)** Find all nine elements of the moment of inertia tensor with respect to the CM of a uniform cuboid (a rectangular brick shape) whose sides are $2a$, $2b$, and $2c$ in the x, y, and z directions and whose mass is M. Explain clearly why you could write down the off-diagonal elements without doing any integration. **(b)** Combine the results of part (a) and Problem 10.24 to find the moment of inertia tensor of the same cuboid with respect to the corner A at (a, b, c). **(c)** What is the angular momentum about A if the cuboid is spinning with angular velocity ω around the edge through A and parallel to the x axis?

(a) As in (10.45)

$$I_{xx} = \int_{-a}^{a} dx \int_{-b}^{b} dy \int_{-c}^{c} dz\, \varrho\, (y^2 + z^2).$$

This is the sum of two terms, each of which is easily evaluated. For example, the first term is

$$\varrho \int_{-a}^{a} dx \int_{-b}^{b} y^2\, dy \int_{-c}^{c} dz = \frac{8}{3}\varrho a b^3 c = \frac{1}{3} M b^2.$$

The second term works the same way, and we conclude that $I_{xx} = \frac{1}{3}M(b^2 + c^2)$, with corresponding results for I_{yy} and I_{zz}. Since the plane $z = 0$ is a plane of reflection symmetry, it follows that the four off-diagonal elements $I_{xz}, \cdots$ that involve z are all zero. Since the other two planes are likewise symmetry planes, we see that all six off-diagonal elements are zero. Thus the whole inertia tensor (which I'll rename $\mathbf{I}^{\text{cm}}$ since it is for rotation about the CM) is

$$\mathbf{I}^{\text{cm}} = \frac{1}{3}M \begin{bmatrix} b^2 + c^2 & 0 & 0 \\ 0 & c^2 + a^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{bmatrix}$$

(b) The "generalized parallel axis theorem" gives the inertia tensor $\mathbf{I}$ with respect to axes through an arbitrary pivot, in terms of the tensor $\mathbf{I}^{\text{cm}}$ referred to parallel axes through the CM. The displacement $\boldsymbol{\Delta} = (\xi, \eta, \zeta)$ from the CM to the corner A is just (a, b, c). Thus, $I_{xx} = I_{xx}^{\text{cm}} + M(b^2 + c^2) =$

$\frac{4}{3}M(b^2 + c^2)$, with similar results for the other two diagonal elements. The off-diagonal elements of $\mathbf{I}^{\rm cm}$ are all zero, so $I_{yz} = -Mbc$, and so on. Thus the whole inertia tensor with respect to A is

$$\mathbf{I} = \frac{1}{3}M\begin{bmatrix} 4(b^2+c^2) & -3ab & -3ac \\ -3ba & 4(c^2+a^2) & -3bc \\ -3ca & -3cb & 4(a^2+b^2) \end{bmatrix}.$$

(c) If $\boldsymbol{\omega}$ is parallel to the x axis, then $\boldsymbol{\omega}$ has components ω, 0, and 0 and, carrying out the indicated matrix multiplication, we find that $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$ has components $\frac{4}{3}M(b^2+c^2)\omega$, $-Mab\omega$, and $-Mac\omega$.

10.27 ⋆⋆⋆ Find the inertia tensor for a uniform, thin hollow cone, such as an ice-cream cone, of mass M, height h, and base radius R, spinning about its pointed end.

Let the mass density of the cone be σ (mass/area) and let us use the cylindrical coordinates ρ and ϕ to specify positions on the cone. (We could equally use z and ϕ, but ρ and ϕ are marginally more convenient.) If we imagine dividing the surface into strips as shown, and the strips into small increments of angle $d\phi$, then the element of area is $dA = (d\rho/\sin\alpha)(\rho\, d\phi)$. The moment about the z axis is then

$$I_{zz} = \int \sigma(x^2+y^2)\,dA = \sigma\int_0^R\int_0^{2\pi}\rho^2\,\frac{\rho\,d\rho\,d\phi}{\sin\alpha} = \frac{\sigma\pi R^4}{2\sin\alpha} \tag{S10.3}$$

since the ϕ integral is just 2π and the ρ integral is $R^4/4$. The area of the cone is $A = \pi R^2/\sin\alpha$. (You can check this by doing the integral $A = \int dA$ if you want.) Thus $\sigma\pi R^2/\sin\alpha = M$, the total mass. Therefore, $I_{zz} = MR^2/2$.

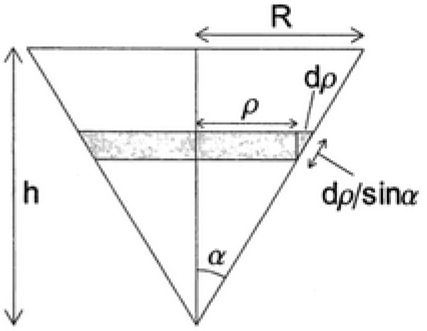


Figure S10.27

The other two moments, I_{xx} and I_{yy}, are equal by rotational symmetry, and

$$I_{xx} = \int \sigma(y^2+z^2)\,dA.$$

The first term here is the same as the second term in (S10.3), where the two terms are equal (by rotational symmetry again). Thus the first term of I_{xx} is just half of I_{zz}. In the second term of I_{xx} we can replace z by $\rho h/R$ (by similar triangles), and we see that the second term is h^2/R^2 times I_{zz}. Putting these together, we find that

$$I_{xx} = I_{yy} = (\tfrac{1}{2} + h^2/R^2)I_{zz} = \tfrac{1}{4}M(R^2+2h^2).$$

Finally, all of the off-diagonal terms are zero by rotational symmetry about the z axis. Therefore

$$\mathbf{I} = \frac{1}{4}M\begin{bmatrix} (R^2+2h^2) & 0 & 0 \\ 0 & (R^2+2h^2) & 0 \\ 0 & 0 & 2R^2 \end{bmatrix}.$$

SECTION 10.4 Principal Axes of Inertia

10.29 ⋆ Prove that if the axes Ox, Oy, and Oz are principal axes of a certain rigid body, then the inertia tensor (with respect to these axes) is diagonal with the principal moments down the diagonal as in (10.66).

That Ox is a principal axis means that if $\boldsymbol{\omega}$ is along Ox then $\mathbf{L}$ is also along Ox. If we think in terms of matrices, this says that if $\boldsymbol{\omega}$ is a column with entries ω, 0, 0, then $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$ is a column whose second and third entries are also zero. This requires that $I_{yx} = I_{zx} = 0$. Similarly, that Oy and Oz are principal axes requires that $I_{xy} = I_{zy} = 0$ and $I_{xz} = I_{yz} = 0$. This leaves $\mathbf{I}$ with I_{xx}, I_{yy}, and I_{zz} down the diagonal and zeroes everywhere else.

10.31 ⋆⋆ Consider an arbitrary rigid body with an axis of rotational symmetry, which we'll call $\hat{\mathbf{z}}$. **(a)** Prove that the axis of symmetry is a principal axis. **(b)** Prove that any two directions $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ perpendicular to $\hat{\mathbf{z}}$ and each other are also principal axes. **(c)** Prove that the principal moments corresponding to these two axes are equal: $\lambda_1 = \lambda_2$.

(a) If the z axis is an axis of rotational symmetry, then we saw in Example 10.1(c) that $I_{xz} = I_{yz} = 0$. Therefore, if $\boldsymbol{\omega} = (0, 0, \omega)$, then $\mathbf{L} = (I_{xz}\omega, I_{yz}\omega, I_{zz}\omega) = (0, 0, I_{zz}\omega)$ and $\hat{\mathbf{z}}$ is a principal axis.
(b) Let us choose any two perpendicular directions in the plane $z = 0$ as our x and y axes. Because $\hat{\mathbf{z}}$ is an axis of rotational symmetry, the xz plane is a plane of reflection symmetry, and this implies that $I_{xy} = 0$ [Example 10.1(b)]. Therefore all off-diagonal elements of $\mathbf{I}$ are zero, and the x and y axes are principal axes.
(c) $I_{xx} = \int \varrho(y^2 + z^2)dV$ and $I_{yy} = \int \varrho(x^2 + z^2)dV$. Now consider a rotation through 90 degrees about the z axis. On the one hand this interchanges y and x (but leaves z unchanged), but, because of the rotational symmetry, it doesn't change the body at all. Therefore, the two integrals are equal and $I_{xx} = I_{yy}$; that is, $\lambda_1 = \lambda_2$.

10.33 ⋆⋆⋆ Here is a good exercise in vector identities and matrices, leading to some important general results: **(a)** For a rigid body made up of particles of mass m_α, spinning about an axis through the origin with angular velocity $\boldsymbol{\omega}$, prove that its total kinetic energy can be written as

$$T = \tfrac{1}{2} \sum m_\alpha[(\omega r_\alpha)^2 - (\boldsymbol{\omega} \cdot \mathbf{r}_\alpha)^2].$$

Remember that $\mathbf{v}_\alpha = \boldsymbol{\omega} \times \mathbf{r}_\alpha$. You may find the following vector identity useful: For any two vectors **a** and **b**,

$$(\mathbf{a} \times \mathbf{b})^2 = a^2 b^2 - (\mathbf{a} \cdot \mathbf{b})^2.$$

(If you use the identity, please prove it.) **(b)** Prove that the angular momentum **L** of the body can be written as

$$\mathbf{L} = \sum m_\alpha[\boldsymbol{\omega} {r_\alpha}^2 - \mathbf{r}_\alpha(\boldsymbol{\omega} \cdot \mathbf{r}_\alpha)].$$

For this you will need the so-called $BAC - CAB$ rule, that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$.
(c) Combine the results of parts (a) and (b) to prove that

$$T = \tfrac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L} = \tfrac{1}{2}\tilde{\boldsymbol{\omega}}\mathbf{I}\boldsymbol{\omega}.$$

Prove both equalities. The last expression is a matrix product; $\boldsymbol{\omega}$ denotes the 3×1 column of numbers ω_x, ω_y, ω_z, the tilde on $\tilde{\boldsymbol{\omega}}$ denotes the matrix transpose (in this case a row), and $\mathbf{I}$ is the moment of inertia tensor. This result is actually quite important; it corresponds to the much more obvious result that for a particle, $T = \frac{1}{2}\mathbf{v} \cdot \mathbf{p}$. **(d)** Show that with respect to the principal axes, $T = \frac{1}{2}(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2)$, as in Equation (10.68).

(a) First

$$T = \tfrac{1}{2}\sum m_\alpha v_\alpha{}^2 = \tfrac{1}{2}\sum m_\alpha(\boldsymbol{\omega} \times \mathbf{r}_\alpha)^2. \tag{S10.4}$$

Now, for any two vectors, $\mathbf{a}\cdot\mathbf{b} = ab\cos\theta$ and $|\mathbf{a} \times \mathbf{b}| = ab\sin\theta$, so $(\mathbf{a}\cdot\mathbf{b})^2 + (\mathbf{a} \times \mathbf{b})^2 = a^2b^2$, which is the required identity. Substituting into (S10.4), we get the required result

$$T = \tfrac{1}{2}\sum m_\alpha[\omega^2 r_\alpha{}^2 - (\boldsymbol{\omega}\cdot\mathbf{r}_\alpha)^2]. \tag{S10.5}$$

(b)

$$\begin{aligned}\mathbf{L} &= \sum m_\alpha \mathbf{r}_\alpha \times \mathbf{v}_\alpha = \sum m_\alpha \mathbf{r}_\alpha \times (\boldsymbol{\omega} \times \mathbf{r}_\alpha)\\ &= \sum m_\alpha[\boldsymbol{\omega}(\mathbf{r}_\alpha\cdot\mathbf{r}_\alpha) - \mathbf{r}_\alpha(\boldsymbol{\omega}\cdot\mathbf{r}_\alpha)].\end{aligned} \tag{S10.6}$$

(c) If we dot Equation (S10.6) with $\boldsymbol{\omega}$ we find

$$\boldsymbol{\omega}\cdot\mathbf{L} = \sum m_\alpha[\omega^2 r_\alpha{}^2 - (\boldsymbol{\omega}\cdot\mathbf{r}_\alpha)^2] = 2T,$$

as given by (S10.5). Therefore,

$$T = \tfrac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L}. \tag{S10.7}$$

The second identity concerns the translation between vector and matrix notations. It may help to have a temporary notation that distinguishes between vectors and matrices. Let us agree to identify matrices with an underscore. Thus if $\mathbf{r} = (x, y, z)$, I shall write

$$\underline{\mathbf{r}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Similarly, I shall write the inertia tensor as $\underline{\mathbf{I}}$ and the angular momentum (column vector) as $\underline{\mathbf{L}} = \underline{\mathbf{I}}\,\underline{\boldsymbol{\omega}}$. To see what the vector dot product looks like in matrix notation, consider the following (as usual, a tilde denotes the transpose):

$$\underline{\tilde{\mathbf{a}}}\,\underline{\mathbf{b}} = [a_x\ a_y\ a_z]\begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = a_x b_x + a_y b_y + a_z b_z = \mathbf{a}\cdot\mathbf{b}\,.$$

In other words, the vector dot product $\mathbf{a}\cdot\mathbf{b}$ is the same as the matrix product $\underline{\tilde{\mathbf{a}}}\,\underline{\mathbf{b}}$ of the row representing $\mathbf{a}$ and the column representing $\mathbf{b}$. Substituting into Eq. (S10.7), we see that $T = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L} = \frac{1}{2}\underline{\tilde{\boldsymbol{\omega}}}\,\underline{\mathbf{I}}\,\underline{\boldsymbol{\omega}}$, as claimed.

(d) With respect to the principal axes $\mathbf{I}$ is a diagonal matrix with the principal moments λ_1, λ_2, and λ_3 on the diagonal. Therefore $\mathbf{L} = \mathbf{I}\,\boldsymbol{\omega}$ is the column with elements $\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3$ and $T = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L} = \frac{1}{2}(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2)$.

SECTION 10.5 Finding the Principal Axes; Eigenvalue Equations

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10.35 ★★ A rigid body consists of three masses fastened as follows: m at $(a, 0, 0)$, $2m$ at $(0, a, a)$, and $3m$ at $(0, a, -a)$. **(a)** Find the inertia tensor $\mathbf{I}$. **(b)** Find the principal moments and a set of orthogonal principal axes.

(a)

$$I_{xx} = \sum m_\alpha (y_\alpha^{\,2} + z_\alpha^{\,2}) = 0 + 2m(a^2 + a^2) + 3m(a^2 + a^2) = 10ma^2$$

$$I_{yy} = \sum m_\alpha (z_\alpha^{\,2} + x_\alpha^{\,2}) = m(a^2) + 2m(a^2) + 3m(a^2) = 6ma^2 = I_{zz}$$

$$I_{yz} = -\sum m_\alpha (y_\alpha z_\alpha) = 0 - 2m(a^2) - 3m(-a^2) = ma^2$$

$$I_{zx} = -\sum m_\alpha (z_\alpha x_\alpha) = 0 + 0 + 0 = 0 = I_{xy}$$

Therefore

$$\mathbf{I} = ma^2 \begin{bmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{bmatrix}$$

(b) For convenience write the eigenvalues of $\mathbf{I}$ as $\lambda = ma^2\lambda'$. Then

$$(\mathbf{I} - \lambda \mathbf{1}) = ma^2 \begin{bmatrix} 10-\lambda' & 0 & 0 \\ 0 & 6-\lambda' & 1 \\ 0 & 1 & 6-\lambda' \end{bmatrix}. \tag{S10.8}$$

This has determinant $\det(\mathbf{I} - \lambda\mathbf{1}) = (ma^2)^3(10-\lambda')(7-\lambda')(5-\lambda')$. Therefore the three eigenvalues (that is, the three principal moments) are

$$\lambda_1 = 10ma^2, \quad \lambda_2 = 7ma^2, \qquad \text{and} \qquad \lambda_3 = 5ma^2$$

To find the corresponding principal axes, we must substitute λ_1, λ_2, and λ_3 in turn into (S10.8) and solve the equation $(\mathbf{I} - \lambda\mathbf{1})\mathbf{a} = 0$. For the first eigenvalue, this gives three equations, $0 = 0$, $-4a_2 + a_3 = 0$, and $a_2 - 4a_3 = 0$. Therefore, $a_2 = a_3 = 0$ and $\mathbf{a} = (a_1, 0, 0)$. Thus for a unit vector in the direction of the first principal axis we can take $\mathbf{e}_1 = (1, 0, 0)$; that is, the first principal axis is the x axis. The other two principal axis are found in the same way to be

$$\mathbf{e}_2 = \tfrac{1}{\sqrt{2}}(0, 1, 1) \qquad \text{and} \qquad \mathbf{e}_3 = \tfrac{1}{\sqrt{2}}(0, 1, -1).$$

In this case, the second axis points toward the mass $2m$, and the third toward the mass $3m$.

10.37 ★★★ A thin, flat, uniform metal triangle lies in the xy plane with its corners at $(1, 0, 0)$, $(0, 1, 0)$, and the origin. Its surface density (mass/area) is $\sigma = 24$. (Distances and masses are measured in unspecified units, and the number 24 was chosen to make the answer come out nicely.) **(a)** Find the triangle's inertia tensor $\mathbf{I}$. **(b)** What are its principal moments and the corresponding axes?

(a) Since all mass is confined to the plane $z = 0$,

$$I_{xx} = \int \sigma y^2 dA = \sigma \int_0^1 dx \int_0^{1-x} y^2 dy = \sigma/12 = 2.$$

I_{yy} is the same, and from Problem 10.23, we know that

$$I_{zz} = I_{xx} + I_{yy} = 4.$$

I_{xz} and I_{yz} are both zero, and

$$I_{xy} = -\int \sigma x y \, dA = -\sigma \int_0^1 x \, dx \int_0^{1-x} y \, dy = -1.$$

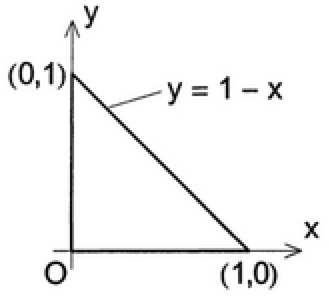

Figure S10.37

Therefore

$$\mathbf{I} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

(b) The characteristic equation is $\det(\mathbf{I} - \lambda\mathbf{1}) = (1-\lambda)(3-\lambda)(4-\lambda) = 0$, so the principal moments are $\lambda_1 = 1$, $\lambda_2 = 3$, and $\lambda_3 = 4$. If we set $\lambda = \lambda_1 = 1$, the equation $(\mathbf{I} - \lambda\mathbf{1})\boldsymbol{\omega} = 0$ implies the three equations $\omega_1 - \omega_2 = 0$, $\omega_1 - \omega_2 = 0$, and $\omega_3 = 0$; thus, the corresponding principal direction is $\mathbf{e}_1 = (1, 1, 0)/\sqrt{2}$. Setting $\lambda = \lambda_2$ and $\lambda = \lambda_3$ in turn, we can similarly find the other two principal directions to be $\mathbf{e}_2 = (1, -1, 0)/\sqrt{2}$ and $\mathbf{e}_3 = (0, 0, 1)$.

SECTION 10.6 Precession of a Top Due to a Weak Torque

10.39 ⋆ Consider a top consisting of a uniform cone spinning freely about its tip at 1800 rpm. If its height is 10 cm and its base radius 2.5 cm, at what angular velocity will it precess?

According to Eq. (10.83), the rate of precession is $\Omega = MgR/(\lambda_3\omega)$, where R is the distance from the tip to the CM of the cone, $R = \frac{3}{4}h$, and λ_3 is the moment of inertia about the cone's axis, $\lambda_3 = \frac{3}{10}Mr^2$ [Eq. (10.59)]. Therefore $\Omega = 5gh/(2r^2\omega) = 21$ rad/s ≈ 200 rpm.

SECTION 10.7 Euler's Equations

10.41 ⋆⋆ Consider a lamina rotating freely (no torques) about a point O of the lamina. Use Euler's equations to show that the component of $\boldsymbol{\omega}$ in the plane of the lamina has constant magnitude. [*Hint:* Use the results of Problems 10.23 and 10.30. According to Problem 10.30, if you choose the direction $\mathbf{e}_3$ normal to the plane of the lamina, $\mathbf{e}_3$ points along a principal axis. Then what you have to prove is that the time derivative of $\omega_1^2 + \omega_2^2$ is zero.]

Choose axes 1, 2, and 3 to be the principal axes through O, with axis 3 perpendicular to the lamina. (By Problem 10.30, this *is* a principal axis.) By Problem 10.23, the corresponding principal moments satisfy $\lambda_1 + \lambda_2 = \lambda_3$. The component of $\boldsymbol{\omega}$ in the plane of the lamina has magnitude squared $\omega_1^2 + \omega_2^2$, whose time derivative is

$$\frac{d}{dt}(\omega_1^2 + \omega_2^2) = \omega_1\dot{\omega}_1 + \omega_2\dot{\omega}_2. \tag{S10.9}$$

Now, by Euler's equations,

$$\lambda_1\dot{\omega}_1 = (\lambda_2 - \lambda_3)\omega_2\omega_3 = -\lambda_1\omega_2\omega_3$$

where the second equality follows from the result of Problem 10.23. Thus,

$$\dot{\omega}_1 = -\omega_2\omega_3 \qquad \text{and similarly} \qquad \dot{\omega}_2 = +\omega_3\omega_1.$$

Substituting into Eq. (S10.9), we conclude that

$$\frac{d}{dt}(\omega_1^2 + \omega_2^2) = -\omega_1\omega_2\omega_3 + \omega_2\omega_3\omega_1 = 0.$$

SECTION 10.8 Euler's Equations with Zero Torque

10.43 ★★ I throw a thin uniform circular disc (think of a frisbee) into the air so that it spins with angular velocity $\boldsymbol{\omega}$ about an axis which makes an angle α with the axis of the disc. **(a)** Show that the magnitude of $\boldsymbol{\omega}$ is constant. [Look at Equation (10.94).] **(b)** Show that as seen by me, the disc's axis precesses around the fixed direction of the angular momentum with angular velocity $\Omega_s = \omega\sqrt{4 - 3\sin^2\alpha}$. (The results of Problems 10.23 and 10.46 will be useful.)

(a) From (10.94) we see that

$$\omega^2 = \omega_o^2\cos^2(\Omega_b t) + \omega_o^2\sin^2(\Omega_b t) + \omega_3^2 = \omega_o^2 + \omega_3^2 = \text{const},$$

since both ω_o and ω_3 are constant.
(b) From (10.94) and (10.95), we see that $\boldsymbol{\omega}$, $\mathbf{L}$ and $\mathbf{e}_3$ lie in a plane and, as seen in the body frame, $\boldsymbol{\omega}$ and $\mathbf{L}$ precess around $\mathbf{e}_3$ with angular frequency (10.93)

$$\Omega_b = \frac{\lambda_3 - \lambda_1}{\lambda_1}\omega_3 = \omega\cos\alpha$$

since (from Problem 10.23) $\lambda_3 = 2\lambda_1$ and $\omega_3 = \omega\cos\alpha$. As seen by me (in the "space" frame), the vector $\mathbf{L}$ is fixed and $\boldsymbol{\omega}$ precesses around $\mathbf{L}$ with frequency Ω_s given by Eq. (10.118) in Problem 10.46

$$\Omega_s = \omega\frac{\sqrt{\lambda_3^2 + (\lambda_1^2 - \lambda_3^2)\sin^2\alpha}}{\lambda_1} = \omega\sqrt{4 - 3\sin^2\alpha}\,.$$

10.45 ★★ Because of the earth's equatorial bulge, its moment about the polar axis is slightly greater that the other two moments, $\lambda_3 = 1.00327\lambda_1$ (but $\lambda_1 = \lambda_2$). **(a)** Show that the free precession described in Section 10.8 should have period 305 days. (As described in the text, the period of this "Chandler wobble" is actually more like 400 days.) **(b)** The angle between the polar axis and $\boldsymbol{\omega}$ is about 0.2 arc seconds. Use Equation (S10.10) from Problem 10.46 (below) to show that as seen from the space frame the period of this wobble should be about a day.

(a) From Equation (10.93) the rate of precession of $\boldsymbol{\omega}$ about the earth's axis $\mathbf{e}_3$ is $\Omega_b = \omega_3(\lambda_1 - \lambda_3)/\lambda_1 = 0.00327\omega_3$. The period of this precession is

$$\tau_b = \frac{2\pi}{\Omega_b} = \frac{1}{0.00327}\frac{2\pi}{\omega_3} = 306 \text{ days}$$

because $2\pi/\omega_3 = 1$ day. This is very nearly, but not quite, the claimed 305 days. The discrepancy is because $2\pi/\omega_3$ is actually 1 sidereal day (the time for one rotation of the earth relative to the stars), and a sidereal day is less than a solar day (what we normally consider to be a day) by about one part in 365. Thus 306 sidereal days are equal to about 305 solar days.

(b) From Fig.10.9 we see that $\tan\alpha = \omega_o/\omega_3$. Since α is tiny ($\alpha = 0.2$ arcseconds $\approx 10^{-6}$ rad), this means that $\omega_o \ll \omega_3$ and hence $\omega = |\boldsymbol{\omega}| \approx \omega_3$. Similarly from (10.95), $L = |\mathbf{L}| \approx L_3 = \lambda_3\omega_3$. Therefore the rate of precession of $\boldsymbol{\omega}$ in the space frame, as given by Eq. (10.96), is $\Omega_s = L/\lambda_1 \approx \lambda_3\omega_3/\lambda_1 \approx \omega_3$, and the corresponding period is $\tau_s = 2\pi/\Omega_s \approx 2\pi/\omega_3 = 1$ day.

10.46 ⋆⋆⋆ We saw in Section 10.8 that in the free precession of an axially symmetric body the three vectors $\mathbf{e}_3$ (the body axis), $\boldsymbol{\omega}$, and $\mathbf{L}$ lie in a plane. As seen in the body frame, $\mathbf{e}_3$ is fixed, and $\boldsymbol{\omega}$ and $\mathbf{L}$ precess around $\mathbf{e}_3$ with angular velocity $\Omega_b = \omega_3(\lambda_1 - \lambda_3)/\lambda_1$. As seen in the space frame $\mathbf{L}$ is fixed and $\boldsymbol{\omega}$ and $\mathbf{e}_3$ precess around $\mathbf{L}$ with angular velocity Ω_s. In this problem you will find three equivalent expressions for Ω_s. **(a)** Argue that $\boldsymbol{\Omega}_s = \boldsymbol{\Omega}_b + \boldsymbol{\omega}$. [Remember that relative angular velocities add like vectors.] **(b)** Bearing in mind that $\boldsymbol{\Omega}_b$ is parallel to $\mathbf{e}_3$ prove that $\Omega_s = \omega\sin\alpha/\sin\theta$ where α is the angle between $\mathbf{e}_3$ and $\boldsymbol{\omega}$ and θ is that between $\mathbf{e}_3$ and $\mathbf{L}$ (see Figure 10.9). **(c)** Thence prove that

$$\Omega_s = \omega\frac{\sin\alpha}{\sin\theta} = \frac{L}{\lambda_1} = \omega\frac{\sqrt{\lambda_3^2 + (\lambda_1^2 - \lambda_3^2)\sin^2\alpha}}{\lambda_1}. \qquad \text{(S10.10)}$$

10.47 ⋆⋆⋆ Imagine that this world is perfectly rigid, uniform, and spherical and is spinning about its usual axis at its usual rate. A huge mountain of mass 10^{-8} earth masses is now added at colatitude 60°, causing the earth to begin the free precession described in Section 10.8. How long will it take the North Pole (defined as the northern end of the diameter along $\boldsymbol{\omega}$) to move 100 miles from its current position? [Take the earth's radius to be 4000 miles.]

Once the mountain has been added, the earth has only one axis of symmetry $\mathbf{e}_3$, which goes through the mountain. In accordance with (10.93), the angular velocity $\boldsymbol{\omega}$ now precesses about the mountain at a rate Ω_b (as seen by us on earth), where (if M denotes the mass of the earth and $m = 10^{-8}M$ that of the mountain)

$$\Omega_b = \frac{\lambda_1 - \lambda_3}{\lambda_1}\omega_3 = \frac{mR^2}{\frac{2}{5}MR^2}\omega_3 = \frac{5m}{2M}\omega\cos\alpha.$$

The distance $d = 100$ mi moved by the pole in time t is $d = (R\sin\alpha)\Omega_b t$. Therefore,

$$t = \frac{d}{(R\sin\alpha)\Omega_b} = \frac{2Md}{5mR\omega\sin\alpha\cos\alpha}$$

or, putting in the numbers (including $\omega = 2\pi/\text{day}$), $t = 3.68 \times 10^5$ days ≈ 1000 years.

SECTION 10.9 Euler Angles

10.49 ⋆⋆ Starting from Equation (10.100) for **L**, verify that L_z is correctly given by Equations (10.102) and (10.103).

Equation (10.100) gives the angular momentum **L** of the top in terms of the basis vectors $\mathbf{e}'_1$, $\mathbf{e}'_2$, and $\mathbf{e}_3$. To find $L_z = \mathbf{L} \cdot \hat{\mathbf{z}}$ we need to know the scalar product of each of these three vectors with $\hat{\mathbf{z}}$. These are easily read off from Fig.10.10 as

$$\mathbf{e}'_1 \cdot \hat{\mathbf{z}} = -\sin\theta, \qquad \mathbf{e}'_2 \cdot \hat{\mathbf{z}} = 0, \qquad \text{and} \qquad \mathbf{e}_3 \cdot \hat{\mathbf{z}} = \cos\theta.$$

According to (10.100)

$$\mathbf{L} = (-\lambda_1 \dot{\phi} \sin\theta)\mathbf{e}'_1 + \lambda_1 \dot{\theta} \mathbf{e}'_2 + \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{e}_3.$$

Taking the dot product of this with $\hat{\mathbf{z}}$, we get

$$L_z = \mathbf{L} \cdot \hat{\mathbf{z}} = \lambda_1 \dot{\phi} \sin^2\theta + \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta$$

which is Eq. (10.102). Finally, using (10.101) we can replace $(\dot{\psi} + \dot{\phi}\cos\theta)$ with L_3/λ_3 to give $L_z = \lambda_1 \dot{\phi} \sin^2\theta + L_3 \cos\theta$, which is (10.103).

SECTION 10.10 Motion of a Spinning Top

10.51 ⋆ Verify that the energy of a symmetric top can be written as $E = \frac{1}{2}\lambda_1 \dot{\theta}^2 + U_{\text{eff}}(\theta)$, where the effective potential energy is as given in (10.114).

From (10.105) we find

$$\begin{aligned} E = T + U &= \tfrac{1}{2}\lambda_1(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \tfrac{1}{2}\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 + MgR\cos\theta \\ &= \tfrac{1}{2}\lambda_1\dot{\theta}^2 + \frac{(L_z - L_3\cos\theta)^2}{2\lambda_1 \sin^2\theta} + \frac{L_3^2}{2\lambda_3} + MgR\cos\theta = \tfrac{1}{2}\lambda_1\dot{\theta}^2 + U_{\text{eff}}(\theta) \end{aligned}$$

where, in moving to the second line, I used (10.104) to replace $\dot{\phi}$ by $(L_z - L_3\cos\theta)/(\lambda_1 \sin^2\theta)$ and (10.101) to replace $(\dot{\psi} + \dot{\phi}\cos\theta)$ by L_3/λ_3. The value of U_{eff} in the last expression is easily read off from the previous one and agrees exactly with Eq. (10.114).

10.53 ⋆⋆ In the discussion of steady precession of a top in Section 10.10, the rates Ω at which steady precession can occur were determined by the quadratic equation (10.110). In particular, we examined this equation for the case that ω_3 is very large. In this case you can write the equation as $a\Omega^2 + b\Omega + c = 0$ where b is very large. **(a)** Verify that when b is very large, the two solutions of this equation are approximately $-c/b$ (which is small) and $-b/a$ (which is large). What precisely does the condition "b is very large" entail? **(b)** Verify that these give the two solutions claimed in (10.111) and (10.112).

(a) We can write Eq. (10.110) for Ω as $a\Omega^2 + b\Omega + c = 0$ if we define $a = \lambda_1\cos\theta$, $b = -\lambda_3\omega_3$, and $c = MgR$. The two solutions for Ω are, of course,

$$\Omega = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 \gg 4ac$, the solution with the minus sign is approximately $\Omega \approx (-b-b)/2a = -b/a$. The solution with a plus sign requires a little more care. With $b^2 \gg 4ac$, we have $\sqrt{b^2-4ac} = b(1-4ac/b^2)^{1/2} \approx b - 2ac/b$, so the second solution is $\Omega \approx (-b+b-2ac/b)/2a = -c/b$. The condition for the validity of these two approximate solutions is just that $b^2 \gg 4ac$. (If $b < 0$ the two solutions reverse roles.)
(b) If we put back the values of a, b, and c, the two solutions become $\Omega = -b/a = \lambda_3\omega_3/(\lambda_1 \cos\theta)$ in agreement with (10.112) and $\Omega = -c/b = MgR/(\lambda_3\omega_3)$ in agreement with (10.111). The condition $b^2 \gg 4ac$ for these to be good approximations translates to $\lambda_3^2\omega_3^2 \gg 4\lambda_1 MgR\cos\theta$; if we assume λ_1 and λ_3 are not too different, this condition is roughly $\frac{1}{2}\lambda_3\omega_3^2 \gg MgR\cos\theta$ (I've dropped a factor of 2) or, even more roughly, that the KE of rotation is much greater than the gravitational PE.

10.55 ⋆⋆⋆ The analysis of the free precession of a symmetric body in Section 10.8 was based on Euler's equations. Obtain the same results using Euler's angles as follows: Since **L** is constant you may as well choose the space axis $\hat{\mathbf{z}}$ so that $\mathbf{L} = L\hat{\mathbf{z}}$. **(a)** Use Equation (10.98) for $\hat{\mathbf{z}}$ to write **L** in terms of the unit vectors $\mathbf{e}'_1$, $\mathbf{e}'_2$, and $\mathbf{e}_3$. **(b)** By comparing this expression with (10.100), obtain three equations for $\dot{\theta}$, $\dot{\phi}$, and $\dot{\psi}$. **(c)** Hence show that θ and $\dot{\phi}$ are constant, and that the rate of precession of the body axis about the space axis $\hat{\mathbf{z}}$ is $\Omega_s = L/\lambda_1$ as in (10.96). **(d)** Using (10.99) show that the angle between $\boldsymbol{\omega}$ and $\mathbf{e}_3$ is constant and that the three vectors **L**, $\boldsymbol{\omega}$, and $\mathbf{e}_3$ are always coplanar.

(a) With $\hat{\mathbf{z}}$ chosen in the direction of **L** and given by (10.98),

$$\mathbf{L} = L\hat{\mathbf{z}} = -L\sin\theta\,\mathbf{e}'_1 + L\cos\theta\,\mathbf{e}_3. \qquad \text{(S10.11)}$$

(b) Comparing this with Eq. (10.100), $\mathbf{L} = (-\lambda_1\dot{\phi}\sin\theta)\mathbf{e}'_1 + \lambda_1\dot{\theta}\mathbf{e}'_2 + \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{e}_3$, we find the following three equations:

$$\lambda_1\dot{\phi}\sin\theta = L\sin\theta, \qquad \dot{\theta} = 0, \qquad \text{and} \qquad \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta) = L\cos\theta. \qquad \text{(S10.12)}$$

(c) The second of Eqs. (S10.12) implies that θ is constant. Cancelling the $\sin\theta$ from the first equation, we see that $\dot{\phi}$ is constant, $\dot{\phi} = L/\lambda_1$. Since $\dot{\phi}$ is the rate at which the body's axis precesses about the space axis $\hat{\mathbf{z}}$, this agrees with the prediction (10.96), $\Omega_s = L/\lambda_1$, for that same rate.
(d) According to (10.99)

$$\boldsymbol{\omega} = (-\dot{\phi}\sin\theta)\mathbf{e}'_1 + (\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{e}_3. \qquad \text{(S10.13)}$$

We already know that the coefficient of $\mathbf{e}'_1$ is constant, and, by the third of Eqs. (S10.12) the coefficient of $\mathbf{e}_3$ is also constant. Therefore, both ω_3 and $|\boldsymbol{\omega}|$ are constant, and so, therefore, is the angle between $\boldsymbol{\omega}$ and $\mathbf{e}_3$.

From Eqs. (S10.11) and (S10.13) it is clear that both **L** and $\boldsymbol{\omega}$ lie in the plane defined by $\mathbf{e}'_1$ and $\mathbf{e}_3$. Therefore, **L**, $\boldsymbol{\omega}$, and $\mathbf{e}_3$ are coplanar at all times.

10.57 ⋆⋆⋆ **(a)** Find the Lagrangian for a symmetric top whose tip is free to slide on a frictionless, horizontal table. For generalized coordinates use the Euler angles (θ, ϕ, ψ) plus X and Y, where (X, Y, Z) is the CM position relative to a fixed point on the table. (Note that the vertical position Z is not an independent coordinate, since $Z = R\cos\theta$.) **(b)** Show that the CM motion of (X, Y) separates completely from the rotational motion. **(c)** Consider the two possible rates of steady precession (10.111) and (10.112) (for given θ and ω_3). How do these differ in the present case from their corresponding values when the tip is held at a fixed pivot?

(a) Writing the KE as the sum of that of the CM plus that of the rotation about the CM, we get

$$\mathcal{L} = \tfrac{1}{2}M(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) + \tfrac{1}{2}\lambda_1^{\text{cm}}(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \tfrac{1}{2}\lambda_3^{\text{cm}}(\dot{\psi} + \dot{\phi}\cos\theta)^2 - MgR\cos\theta$$

where λ_1^{cm} and λ_3^{cm} are the principal moments for rotation about the CM and $\dot{Z} = R\dot{\theta}\sin\theta$. [Compare Eq. (10.106).] By the parallel axis theorem (Problem 10.24) the moments about the CM are related to those about the tip as follows:

$$\lambda_3^{\text{cm}} = \lambda_3^{\text{tip}} \qquad \text{and} \qquad \lambda_1^{\text{cm}} = \lambda_1^{\text{tip}} - MR^2. \tag{S10.14}$$

(b) Clearly the motion of X and Y is independent of that of θ, ϕ, and ψ, and the position (X, Y) moves like a free particle with $\dot{X}$ and $\dot{Y}$ constant.

(c) In steady precession θ is constant, so $\dot{Z} = 0$. Therefore the angular part of the Lagrangian is the same as before, in (10.106), except for the replacement of the moments as in (S10.14), and the precession rates are given by the same formulas as before, but with the new moments. The rate of the fast precession is given by (10.112) as $\Omega_{\text{fast}} = \lambda_3\omega_3/(\lambda_1\cos\theta)$. In view of the relations (S10.14), you can see that $\Omega_{\text{fast}}^{\text{cm}} > \Omega_{\text{fast}}^{\text{tip}}$ (for given θ and ω_3). That is, the fast precession is even faster when the tip is free to move.

The rate of the slow precession is given by (10.111) as $\Omega_{\text{slow}} \approx MgR/(\lambda_3\omega_3)$, and, since $\lambda_3^{\text{cm}} = \lambda_3$, this is the same whether the tip is fixed or not. However, (10.111) is only an approximation. If we keep the next term in that approximation, we find

$$\Omega_{\text{slow}} \approx \frac{MgR}{\lambda_3\omega_3}\left[1 + \frac{\lambda_1 MgR\cos\theta}{(\lambda_3\omega_3)^2}\right].$$

(To check this, do Problem 10.53 and keep one extra term.) Thus the slow precession is a little slower when the tip is free to move.

CHAPTER 11

Coupled Oscillators and Normal Modes

PROBLEMS AND SOLUTIONS

SECTION 11.1 Two Masses and Three Springs

11.1 ⋆ In discussing the two carts of Figure 11.1, I mentioned that it is simplest to assume that when the two carts are in equilibrium the lengths L_1, L_2, L_3 of the three springs are equal to their natural, unstretched lengths l_1, l_2, l_3. However, this assumption is not needed, and the three springs could all be in tension (or compression) at the equilibrium position. **(a)** Find the relations among these six lengths (and the three spring constants k_1, k_2, k_3) required for the two carts to be in equilibrium. **(b)** Show that the net force on either cart is exactly as given in Equation (11.2), irrespective of how L_1, L_2, L_3 compare with l_1, l_2, l_3, just as long as x_1 and x_2 are measured from the carts' equilibrium positions.

(a) When the two carts are in equilibrium, the tensions in the three springs must be equal, so

$$k_1(L_1 - \ell_1) = k_2(L_2 - \ell_2) = k_3(L_3 - \ell_3).$$

(b) The net force on cart 1 is

$$F(\text{on cart } 1) = -k_1(x_1 + L_1 - \ell_1) + k_2(x_2 - x_1 + L_2 - \ell_2) = -k_1x_1 + k_2(x_2 - x_1)$$

because the term involving $L_1 - \ell_1$ cancels that involving $L_2 - \ell_2$. This is exactly the expression (11.1), derived on the assumption that the springs were unstretched in equilibrium. The force on cart 2 works similarly.

SECTION 11.2 Identical Springs and Equal Masses

11.3 ⋆ Find the normal frequencies for the system of two carts and three springs shown in Figure 11.1, for arbitrary values of m_1 and m_2 and of k_1, k_2, and k_3. Check that your answer is correct for the case that $m_1 = m_2$ and $k_1 = k_2 = k_3$.

Using the matrices **M** and **K** given in Eq. (11.5), we find

$$(\mathbf{K} - \omega^2\mathbf{M}) = \begin{bmatrix} k_1 + k_2 - m_1\omega^2 & -k_2 \\ -k_2 & k_2 + k_3 - m_2\omega^2 \end{bmatrix}$$

[Compare Eq. (11.14).] Hence

$$\det(\mathbf{K} - \omega^2\mathbf{M}) = m_1m_2\omega^4 - [m_1(k_2 + k_3) + m_2(k_1 + k_2)]\omega^2 + (k_1k_2 + k_1k_3 + k_2k_3)$$

This is zero when

$$\omega^2 = \frac{1}{2m_1m_2}\Big\{m_1(k_2+k_3)+m_2(k_1+k_2)$$
$$\pm\sqrt{m_1^2(k_2+k_3)^2+m_2^2(k_1+k_2)^2+2m_1m_2(k_2^2-k_1k_2-k_1k_3-k_2k_3)}\Big\}$$

Therefore, these are the two normal frequencies (squared).

If $m_1 = m_2 = m$ and $k_1 = k_2 = k_3 = k$, these reduce to $\omega^2 = (k/m)(2 \pm 1)$ in agreement with Eq. (11.15).

11.5 ★★ **(a)** Find the normal frequencies, ω_1 and ω_2, for the two carts shown in Figure 11.15, assuming that $m_1 = m_2$ and $k_1 = k_2$. **(b)** Find and describe the motion for each of the normal modes in turn.

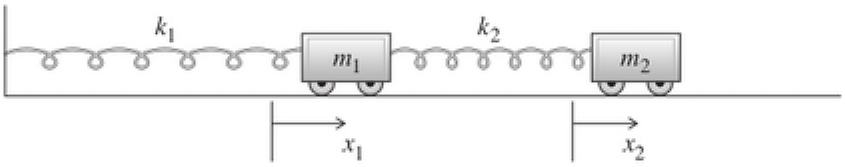

Figure 11.15 Problems 11.5 and 11.6.

(a) The quickest way to find the equation of motion for the system of Fig.11.15 is to set $k_3 = 0$ in Fig.11.1. With $m_1 = m_2$ and $k_1 = k_2$ as well, the mass and spring-constant matrices are given by Eq. (11.5) as

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \qquad \text{and} \qquad \mathbf{K} = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}.$$

If we define $\omega_o = \sqrt{k/m}$ and hence $k = m\omega_o^2$, the characteristic equation becomes

$$\det(\mathbf{K} - \omega^2\mathbf{M}) = m^2(\omega^4 - 3\omega_o^2\omega^2 + \omega_o{}^4) = 0,$$

so the normal frequencies are given by $\omega^2 = \omega_o^2(3 \pm \sqrt{5})/2$.

(b) If we substitute $\omega = \omega_1 = \omega_o\sqrt{(3-\sqrt{5})/2}$, the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$ yields $a_2 = a_1(1+\sqrt{5})/2 = 1.62a_1$. Thus the first normal mode has the form

$$x_1 = A\cos(\omega_1 t - \delta) \qquad \text{and} \qquad x_2 = 1.62A\cos(\omega_1 t - \delta).$$

where A and δ are arbitrary constants. In the first mode the two carts oscillate in phase, the second one with the larger amplitude. Similarly, in the second mode, we find $a_2 = a_1(1-\sqrt{5})/2 = -0.62a_1$ and so

$$x_1 = A\cos(\omega_2 t - \delta) \qquad \text{and} \qquad x_2 = -0.62A\cos(\omega_2 t - \delta).$$

where (in general) A and δ are different constants. In this mode the two carts move 180° out of phase, and cart 2 has the smaller amplitude.

11.7 ★★ [Computer] The most general motion of the two carts of Section 11.2 is given by (11.21), with the constants A_1, A_2, δ_1, and δ_2 determined by the initial conditions. **(a)** Show that (11.21) can be rewritten as

$$\mathbf{x}(t) = (B_1 \cos\omega_1 t + C_1 \sin\omega_1 t)\begin{bmatrix}1\\1\end{bmatrix} + (B_2 \cos\omega_2 t + C_2 \sin\omega_2 t)\begin{bmatrix}1\\-1\end{bmatrix}.$$

This form is usually a little more convenient for matching to given initial conditions. **(b)** If the carts are released from rest at positions $x_1(0) = x_2(0) = A$, find the coefficients B_1, B_2, C_1, and C_2 and plot $x_1(t)$ and $x_2(t)$. Take $A = \omega_1 = 1$ and $0 \le t \le 30$ for your plots. **(c)** Same as part (b), except that $x_1(0) = A$ but $x_2(0) = 0$.

(a) Since $A\cos(\omega t - \delta) = A\cos\omega t\cos\delta + A\sin\omega t\sin\delta = B\cos\omega t + C\sin\omega t$, we can immediately rewrite (11.21) in the given form.
(b) From the given form, you can see that

$$\mathbf{x}(0) = \begin{bmatrix}B_1 + B_2\\B_1 - B_2\end{bmatrix} \quad \text{and} \quad \dot{\mathbf{x}}(0) = \begin{bmatrix}\omega_1 C_1 + \omega_2 C_2\\\omega_1 C_1 - \omega_2 C_2\end{bmatrix}, \tag{S11.1}$$

but we are given that $x_1(0) = x_2(0) = A$ and $\dot{x}_1(0) = \dot{x}_2(0) = 0$. Solving for the coefficients in (S11.1), we find that

$$B_1 = A \quad \text{and} \quad B_2 = C_1 = C_2 = 0.$$

Substituting into the given form, you can see that this solution happens to be the first normal mode, a result we could have anticipated since the system started out with $x_1 = x_2$ and $\dot{x}_1 = \dot{x}_2$.

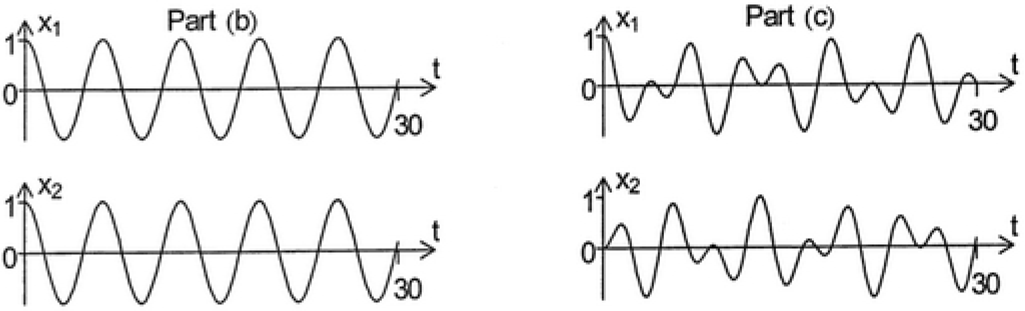

Figure S11.7

(c) In this case the initial conditions are $x_1(0) = A$ and $x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$. Putting these into the left side of (S11.1), you will find that

$$B_1 = B_2 = A/2 \quad \text{and} \quad C_1 = C_2 = 0.$$

This is an equal mixture of the two normal modes and produces the graphs shown in the right-hand figures above.

11.9 ⋆⋆ **(a)** Write down the equations of motion (11.2) for the equal-mass carts of Section 11.2 with three identical springs. Show that the change of variables to the normal coordinates $\xi_1 = \frac{1}{2}(x_1 + x_2)$ and $\xi_2 = \frac{1}{2}(x_1 - x_2)$ leads to uncoupled equations for ξ_1 and ξ_2. **(b)** Solve for ξ_1 and ξ_2 and hence write down the general solution for x_1 and x_2. (Notice how very simple this procedure is, once you have guessed what the normal coordinates are. For a simple symmetric system like this, you can sometimes guess the form of ξ_1 and ξ_2 by considering the symmetry — especially once you have some experience working with normal modes.)

(a) With identical masses and springs, the two equations of motion (11.2) are

$$m\ddot{x}_1 = -2kx_1 + kx_2 \quad \text{and} \quad m\ddot{x}_2 = kx_1 - 2kx_2.$$

If we add these two equations and define $\xi_1 = \frac{1}{2}(x_1 + x_2)$, we find that $m\ddot{\xi}_1 = -k\xi_1$. Similarly, if we subtract the second equation from the first and define $\xi_2 = \frac{1}{2}(x_1 - x_2)$, we get $m\ddot{\xi}_2 = -3k\xi_2$. These equations for ξ_1 and ξ_2 are uncoupled, as claimed.
(b) The general solutions of the equations of motion for ξ_1 and ξ_2 are $\xi_1 = A_1\cos(\omega_1 t - \delta_1)$ and $\xi_2 = A_2\cos(\omega_2 t - \delta_2)$, where A_1, A_2, δ_1 and δ_2 are arbitrary constants, $\omega_1 = \sqrt{k/m}$, and $\omega_2 = \sqrt{3k/m}$. Therefore

$$x_1 = \xi_1 + \xi_2 = A_1\cos(\omega_1 t - \delta_1) + A_2\cos(\omega_2 t - \delta_2)$$

and

$$x_2 = \xi_1 - \xi_2 = A_1\cos(\omega_1 t - \delta_1) - A_2\cos(\omega_2 t - \delta_2)$$

in agreement with Eq. (11.21).

11.11 ★★★ **(a)** Write down the equations of motion corresponding to (11.2) for the equal-mass carts of Section 11.2 with three identical springs, but with each cart subject to a linear resistive force $-b\mathbf{v}$ (same coefficient b for both carts) and with a driving force $F(t) = F_o\cos\omega t$ applied to cart 1. **(b)** Show that if you change variables to the normal coordinates $\xi_1 = \frac{1}{2}(x_1 + x_2)$ and $\xi_2 = \frac{1}{2}(x_1 - x_2)$, the equations of motion for ξ_1 and ξ_2 are uncoupled. **(c)** Using the methods of Section 5.5, write down the general solutions. **(d)** Assuming that $\beta = b/2m \ll \omega_o$, show that ξ_1 resonates when $\omega \approx \omega_o = \sqrt{k/m}$ and likewise ξ_2 when $\omega \approx \sqrt{3}\omega_o$. **(e)** Prove, on the other hand, that if both carts are driven in phase with the same force $F_o\cos\omega t$, only ξ_1 shows a resonance. Explain.

(a) As in Section 5.4, let us define $\beta = b/2m$ and $\omega_o^2 = k/m$. Then the equations of motion are

$$\begin{aligned} \ddot{x}_1 &= -2\beta\dot{x}_1 - 2\omega_o^2 x_1 + \omega_o^2 x_2 + (F_o/m)\cos\omega t \\ \ddot{x}_2 &= -2\beta\dot{x}_2 + \omega_o^2 x_1 - 2\omega_o^2 x_2. \end{aligned} \tag{S11.2}$$

(b) If you take first the sum and then the difference of these two equations, you should get

$$\ddot{\xi}_1 + 2\beta\dot{\xi}_1 + \omega_o^2\xi_1 = f_o\cos\omega t \qquad \text{and} \qquad \ddot{\xi}_2 + 2\beta\dot{\xi}_2 + 3\omega_o^2\xi_2 = f_o\cos\omega t, \tag{S11.3}$$

where $f_o = F_o/(2m)$. These two equations for ξ_1 and ξ_2 are uncoupled, as advertised.
(c) The equation for ξ_1 is the same as Eq. (5.57) for a single driven oscillator and can be solved in the same way. We write $\xi_1(t) = \text{Re}\,\zeta_1(t)$ and try a solution of the form $\zeta_1(t) = C_1 e^{i\omega t}$. Substituting this guess into the first of Eqs. (S11.3), we find that it is a solution provided $(-\omega^2 + 2i\beta\omega + \omega_o^2)C_1 = f_o$. Thus we have found a solution with $C_1 = A_1 e^{-i\delta}$ where A_1 is given by the resonance formula (5.64)

$$A_1^2 = \frac{f_o^2}{(\omega^2 - \omega_o^2)^2 + 4\beta^2\omega^2} \tag{S11.4}$$

and the phase shift δ_1 is given by (5.65). To this particular solution we can add any solution of the corresponding homogeneous equation, and the general solution has the form

$$\xi_1(t) = A_1\cos(\omega t - \delta_1) + B_1 e^{-\beta t}\cos(\omega_1 t - \delta_1')$$

where B_1 and δ_1' are arbitrary constants, determined by the initial conditions, and the frequency of the transient is $\omega_1 = \sqrt{\omega_o^2 - \beta^2}$. The second of Eqs. (S11.3) can be solved in exactly the same way giving an analogous expression for $\xi_2(t)$, the only important differences being that the amplitude of the particular solution is given by

$$A_2^2 = \frac{f_o^2}{(\omega^2 - 3\omega_o^2)^2 + 4\beta^2\omega^2} \tag{S11.5}$$

and the frequency of the transient is $\omega_2 = \sqrt{3\omega_o^2 - \beta^2}$.

(d) If $\beta \ll \omega_o$, it is evident from Eq. (S11.4) that A_1 has a sharp maximum close to $\omega^2 = \omega_o^2$ and, from (S11.5) A_2 has a sharp maximum close to $\omega^2 = 3\omega_o^2$.

(e) If both carts are driven in phase with the same driving force, then both of the equations of motion (S11.2) contain the term $(F_o/m)\cos\omega t$. Therefore the equation in (S11.3) for ξ_1 has the same form as before (except that $f_o = F_o/m$ now) but the equation for ξ_2, which comes from taking the difference of the two equations (S11.2), contains no driving force at all. Therefore, ξ_1 shows the same resonance as before (with twice the amplitude), but ξ_2 shows none.

The reason for this difference is easy to see. The mode represented by ξ_1 has the two carts oscillating exactly in phase, while in that of ξ_2 they oscillate 180° out of phase. By driving them in phase, we are driving mode 1 but not mode 2.

SECTION 11.3 Two Weakly Coupled Oscillators

11.13 ★★★ [Computer] Consider the two carts of Section 11.3, coupled by a weak spring and subject to a resistive force $-b\mathbf{v}$ (same force for each cart). **(a)** Write down the equations of motion for x_1 and x_2 in the form (11.2) and show that if you change to the normal coordinates $\xi_1 = \frac{1}{2}(x_1 + x_2)$ and $\xi_2 = \frac{1}{2}(x_1 - x_2)$ the equations of motion for ξ_1 and ξ_2 are uncoupled. **(b)** Solve for $\xi_1(t)$ and $\xi_2(t)$ assuming that the dissipative coefficient b is small ("underdamped motion," as in Section 5.4), and hence write down the general solution for $x_1(t)$ and $x_2(t)$. **(c)** Suppose that cart 1 is held at $x_1 = A$ and cart 2 at $x_2 = 0$, and they are released from rest at $t = 0$. Find and plot the two positions as functions of t for $0 \le t \le 80$, using the values $A = k = m = 1$, $k_2 = 0.2$, and $b = 0.04$. (In matching the initial conditions, take advantage of the fact that $b \ll 1$, and use a suitable computer program to make the plot.) Comment on your plots.

(a) The two equations of motion in the form (11.2) are

$$m\ddot{x}_1 = -b\dot{x}_1 - (k + k_2)x_1 + k_2x_2 \qquad \text{and} \qquad m\ddot{x}_2 = -b\dot{x}_2 + k_2x_1 - (k + k_2)x_2. \tag{S11.6}$$

If we add these two equations, we find that

$$m\ddot{\xi}_1 = -b\dot{\xi}_1 - k\xi_1 \quad \text{or} \quad \ddot{\xi}_1 + 2\beta\dot{\xi}_1 + \omega_o^2\xi_1 = 0$$

where I have introduced $\beta = b/2m$ as usual and $\omega_o^2 = k/m$. Similarly if we subtract the second of Eqs. (S11.6) from the first we get

$$\ddot{\xi}_2 + 2\beta\dot{\xi}_2 + \omega_o'^2\xi_2 = 0$$

where $\omega_o'^2 = (k + 2k_2)/m$. The equations for ξ_1 and ξ_2 are visibly uncoupled.

(b) The solutions of these two equations are given in (5.38), though they are slightly more convenient in the equivalent form

$$\xi_1 = e^{-\beta t}(B_1\cos\omega_1 t + C_1\sin\omega_1 t) \qquad \text{and} \qquad \xi_2 = e^{-\beta t}(B_2\cos\omega_2 t + C_2\sin\omega_2 t),$$

where $\omega_1 = \sqrt{\omega_o^2 - \beta^2}$ and $\omega_2 = \sqrt{\omega_o'^2 - \beta^2}$.

(c) The initial conditions imply that $\xi_1(0) = \xi_2(0) = A/2$, and hence that $B_1 = B_2 = A/2$. The initial velocities are a little more complicated since $\dot{\xi}_1(0)$ and $\dot{\xi}_2(0)$ both contain two terms, one proportional to β (from differentiating the factor $e^{-\beta t}$). However, to the extent that $\beta \ll \omega_o$, we can ignore these factors and the initial conditions tell us simply that $C_1 = C_2 = 0$. Thus

$$x_1 = \xi_1 + \xi_2 = (A/2)e^{-\beta t}(\cos\omega_1 t + \cos\omega_2 t) \quad\text{and}\quad x_2 = \xi_1 - \xi_2 = (A/2)e^{-\beta t}(\cos\omega_1 t - \cos\omega_2 t).$$

Putting in the given numbers we get the two plots shown below, where you can see how the energy is transferred back and forth between the two carts and, at the same time, slowly dissipated.

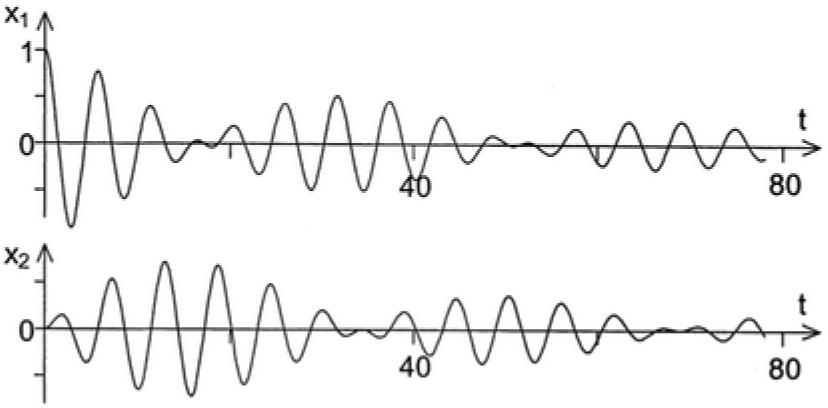

Figure S11.13

SECTION 11.4 Lagrangian Approach: The Double Pendulum

11.15 ⋆⋆ Write down the exact Lagrangian (good for all angles) for the double pendulum of Figure 11.9 and find the corresponding equations of motion. Show that they reduce to Equations (11.41) and (11.42) if both angles are small.

Combining (11.38) and (11.37), we find

$$\begin{aligned}\mathcal{L} &= \tfrac12(m_1+m_2)L_1^2\dot\phi_1^2 + m_2L_1L_2\dot\phi_1\dot\phi_2\cos(\phi_1-\phi_2) + \tfrac12 m_2L_2^2\dot\phi_2^2 \\ &\quad - (m_1+m_2)gL_1(1-\cos\phi_1) - m_2gL_2(1-\cos\phi_2).\end{aligned}$$

Since

$$\frac{\partial\mathcal{L}}{\partial\dot\phi_1} = (m_1+m_2)L_1^2\dot\phi_1 + m_2L_1L_2\dot\phi_2\cos(\phi_1-\phi_2)$$

the ϕ_1 equation is

$$\begin{aligned}&(m_1+m_2)L_1^2\ddot\phi_1 + m_2L_1L_2\ddot\phi_2\cos(\phi_1-\phi_2) - m_2L_1L_2\dot\phi_2(\dot\phi_1-\dot\phi_2)\sin(\phi_1-\phi_2) \\ &\qquad = -m_2L_1L_2\dot\phi_1\dot\phi_2\sin(\phi_1-\phi_2) - (m_1+m_2)gL_1\sin\phi_1.\end{aligned}$$

Since the first term on the right cancels the second to last term on the left, this simplifies slightly to

$$\begin{aligned}&(m_1+m_2)L_1^2\ddot\phi_1 + m_2L_1L_2\ddot\phi_2\cos(\phi_1-\phi_2) + m_2L_1L_2\dot\phi_2^2\sin(\phi_1-\phi_2) \\ &\qquad = -(m_1+m_2)gL_1\sin\phi_1.\end{aligned}$$

In the same way the ϕ_2 equation is

$$\begin{aligned}&m_2L_1L_2\ddot\phi_1\cos(\phi_1-\phi_2) + m_2L_2^2\ddot\phi_2 - m_2L_1L_2\dot\phi_1^2\sin(\phi_1-\phi_2) \\ &\qquad = -m_2gL_2\sin\phi_2.\end{aligned}$$

If both ϕ_1 and ϕ_2 are small, then $\cos(\phi_1 - \phi_2) \approx 1$ and $\sin\phi \approx \phi$ (for either angle). Also the last term on the left of both equations is the product of three small quantities and can be neglected. With these approximations, the two equations here reduce to precisely Equations (11.41) and (11.42).

11.17 ⋆⋆ **(a)** Find the normal frequencies and modes of the double pendulum of Figure 11.9, given that $m_1 = 8m$, $m_2 = m$, and $L_1 = L_2 = L$. **(b)** Find the actual motion $[\phi_1(t), \phi_2(t)]$ if the pendulum is released from rest with $\phi_1 = 0$ and $\phi_2 = \alpha$. Is this motion periodic?

(a) From (11.44) we find

$$\mathbf{M} = mL^2 \begin{bmatrix} 9 & 1 \\ 1 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{K} = mL^2\omega_o^2 \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}.$$

whence

$$\mathbf{K} - \omega^2\mathbf{M} = mL^2 \begin{bmatrix} 9(\omega_o^2 - \omega^2) & -\omega^2 \\ -\omega^2 & (\omega_o^2 - \omega^2) \end{bmatrix} \tag{S11.7}$$

and $\det(\mathbf{K} - \omega^2\mathbf{M}) = m^2L^4(4\omega^2 - 3\omega_o^2)(2\omega^2 - 3\omega_o^2)$. Thus the normal frequencies are

$$\omega_1^2 = \tfrac{3}{4}\omega_o^2 \qquad \text{and} \qquad \omega_2^2 = \tfrac{3}{2}\omega_o^2.$$

If we put $\omega = \omega_1$ in Eq. (S11.7), the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$ implies that $3a_1 = a_2$. If we put $\omega = \omega_2$, we find $3a_1 = -a_2$. Thus,

$$\text{for the first mode, } \mathbf{a} = A_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ and for the second, } \mathbf{a} = A_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix},$$

where A_1 and A_2 are arbitrary complex constants. In the first mode the pendulums oscillate in phase with the amplitude of the lower one three times that of the upper. In the second mode the pendulums oscillate exactly out of phase with the amplitude of the lower one three times that of the upper.
(b) The general solution is a sum of the two modes given above. The initial conditions that $\phi_1(0) = 0$ while $\phi_2(0) = \alpha$ and $\dot\phi_1 = \dot\phi_2 = 0$ require that

$$\operatorname{Re}\left\{A_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + A_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix}\right\} = \begin{bmatrix} 0 \\ \alpha \end{bmatrix} \qquad \text{and} \qquad \operatorname{Re}\left\{i\omega_1 A_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + i\omega_2 A_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix}\right\} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which imply that $A_1 = -A_2 = \alpha/6$. Putting in the time dependent exponentials and taking the real part, we conclude that

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \frac{\alpha}{6}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix} \cos\omega_1 t - \begin{bmatrix} 1 \\ -3 \end{bmatrix} \cos\omega_2 t\right\}.$$

Because $\omega_2 = \sqrt{2}\omega_1$, this is not periodic.

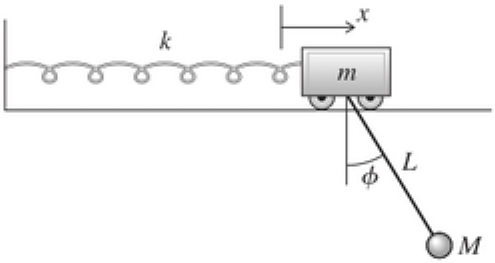

Figure 11.18 Problem 11.19

11.19 ⋆⋆⋆ A simple pendulum (mass M and length L) is suspended from a cart (mass m) that can oscillate on the end of a spring of force constant k, as shown in Figure 11.18. **(a)** Assuming that the angle ϕ remains small, write down the system's Lagrangian and the equations of motion for x and ϕ. **(b)** Assuming that $m = M = L = g = 1$ and $k = 2$ (all in appropriate units) find the normal frequencies, and for each normal frequency find and describe the motion of the corresponding normal mode.

(a) The PE (measured from the equilibrium positions) is

$$U = \tfrac{1}{2}kx^2 + MgL(1 - \cos\phi) \approx \tfrac{1}{2}kx^2 + \tfrac{1}{2}MgL\phi^2.$$

The KE is a bit trickier because the velocity of the bob M is the vector sum of the velocity $\dot{x}$ of the support and $L\dot{\phi}$ of the bob relative to the support. In general these are in different directions, but as long as ϕ remains small they are essentially parallel, and the KE is

$$T \approx \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}M(\dot{x} + L\dot{\phi})^2 = \tfrac{1}{2}(m + M)\dot{x}^2 + ML\dot{x}\dot{\phi} + \tfrac{1}{2}ML^2\dot{\phi}^2.$$

From these you can write down the Lagrangian $\mathcal{L}$ and the two Lagrange equations. The x equation is

$$(m + M)\ddot{x} + ML\ddot{\phi} = -kx$$

and the ϕ equation is

$$ML\ddot{x} + ML^2\ddot{\phi} = -MgL\phi.$$

We can write these as a single matrix equation, $\mathbf{M}\ddot{\mathbf{q}} = -\mathbf{K}\mathbf{q}$, if we define

$$\mathbf{q} = \begin{bmatrix} x \\ \phi \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m + M & ML \\ ML & ML^2 \end{bmatrix}, \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} k & 0 \\ 0 & MgL \end{bmatrix}.$$

(b) With the given values, the matrix $(\mathbf{K} - \omega^2\mathbf{M})$ is

$$\mathbf{K} - \omega^2\mathbf{M} = \begin{bmatrix} 2 - 2\omega^2 & -\omega^2 \\ -\omega^2 & 1 - \omega^2 \end{bmatrix}$$

which has determinant $\det(\mathbf{K} - \omega^2\mathbf{M}) = 2(1 - \omega^2)^2 - \omega^4 = \omega^4 - 4\omega^2 + 2$. The normal frequencies are the zeros of this determinant and are

$$\omega_1 = \sqrt{2 - \sqrt{2}} = 0.77 \qquad \text{and} \qquad \omega_2 = \sqrt{2 + \sqrt{2}} = 1.85.$$

The motion in each corresponding mode is given by the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$. For the first mode, this gives $a_2 = \sqrt{2}a_1$, so the cart and bob oscillate in phase (both moving to the right and then both moving to the left), with the bob's amplitude (of motion relative to the cart) $\sqrt{2}$ times bigger than the cart's. For the second mode, $a_2 = -\sqrt{2}a_1$, so the cart and bob oscillate exactly out of phase, again with the amplitude of the bob equal to $\sqrt{2}$ times that of the cart.

SECTIONS 11.5 and 11.6 The General Case & Three Coupled Pendulums

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11.21 ⋆ Verify that if $U = \frac{1}{2}\sum_j \sum_k K_{jk}q_jq_k$, where the coefficients K_{jk} are all constant and satisfy $K_{ij} = K_{ji}$, then $\partial U/\partial q_i = \sum_j K_{ij}q_j$, as claimed in Equation (11.58).

Let us focus our attention on a single coordinate q_i. The double sum that defines U contains n^2 terms because both j and k run from 1 to n. We can divide these n^2 terms into four groups: first, a single term with $j = k = i$; second, $(n-1)$ terms with $j = i$ but $k \neq i$; third, $(n-1)$ terms with $j \neq i$ but $k = i$; and finally $(n-1)^2$ terms with both j and k not equal to i. This gives

$$U = \tfrac{1}{2}K_{ii}{q_i}^2 + \tfrac{1}{2}\sum_{k\neq i} K_{ik}q_iq_k + \tfrac{1}{2}\sum_{j\neq i} K_{ji}q_jq_i + \tfrac{1}{2}\sum_{j\neq i}\sum_{k\neq i} K_{jk}q_jq_k \tag{S11.8}$$

$$= \tfrac{1}{2}K_{ii}{q_i}^2 + q_i\sum_{j\neq i} K_{ij}q_j + \tfrac{1}{2}\sum_{j\neq i}\sum_{k\neq i} K_{jk}q_jq_k. \tag{S11.9}$$

Here, in passing from (S11.8) to (S11.9), I replaced the dummy index k in the second term on the right of (S11.8) by j and used the fact that $K_{ji} = K_{ij}$ in the third term. We can now differentiate with respect to q_i to give [note that the whole third sum in (S11.9) is independent of q_i, so its derivative is zero]

$$\frac{\partial U}{\partial q_i} = K_{ii}q_i + \sum_{j\neq i} K_{ij}q_j = \sum_j K_{ij}q_j.$$

11.23 ⋆⋆ Equation (11.73) gives the three normal frequencies of three coupled pendulums in natural units with $L = m = 1$. We have already seen that the value of ω_1^2 in arbitrary units is g/L. Find the values of ω_2^2 and ω_3^2 in arbitrary units. [*Hint:* Start by considering the quantity $\omega_2^2 - \omega_1^2$.]

Equation (11.73) asserts that $\omega_1^2 = g$, $\omega_2^2 = g + k$, and $\omega_3^2 = g + 3k$ in a system of units where $L = m = 1$, that is, a system in which the unit of length is L and the unit of mass is m. Taking the difference of the first two results of (11.73), we see that $\omega_2^2 - \omega_1^2 = k$ or

$$\frac{k}{\omega_2^2 - \omega_1^2} = 1 \tag{S11.10}$$

in a system of units where the unit of mass is m. Now, as you can easily check, the dimension of k is $[k] = MT^{-2}$ while that of ω is $[\omega] = T^{-1}$. Therefore, $[k/(\omega_2^2 - \omega_1^2)] = M$, and (S11.10) asserts that $k/(\omega_2^2 - \omega_1^2)$ is a mass whose value is 1 in our special units where $m = 1$. Therefore its value in any system is

$$\frac{k}{\omega_2^2 - \omega_1^2} = m$$

and $\omega_2^2 = \omega_1^2 + k/m = g/L + k/m$. Similarly $\omega_3^2 = g/L + 3k/m$.

11.25 ⋆⋆ Consider a system of carts and springs like that in Figure 11.1 except that there are *three* equal-mass carts and *four* identical springs. Solve for the three normal frequencies, and find and describe the motion in the corresponding normal modes.

The mass and spring constant matrices are

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \qquad \text{and} \qquad \mathbf{K} = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix}$$

from which we find

$$\det(K - \omega^2 M) = m^3(2\omega_o^2 - \omega^2)\left[(2-\sqrt{2})\omega_o^2 - \omega^2\right]\left[(2+\sqrt{2})\omega_o^2 - \omega^2\right]$$

where I have introduced the shorthand, $\omega_o = \sqrt{k/m}$. Thus the normal frequencies are

$$\omega_1 = \omega_o\sqrt{2-\sqrt{2}}, \qquad \omega_2 = \omega_o\sqrt{2}, \qquad \text{and} \qquad \omega_3 = \omega_o\sqrt{2+\sqrt{2}}$$

In the first mode, the eigenvector **a** has $a_1 = a_3 = a_2/\sqrt{2}$, so all three carts oscillate in phase, with the middle cart's amplitude $\sqrt{2}$ bigger than the outer two. In the second, $a_2 = 0$, while $a_1 = -a_3$, so the middle cart is stationary, while the first and third oscillate exactly out of phase. In the third mode, $a_1 = a_3 = -a_2/\sqrt{2}$, so the first and third carts oscillate in phase, while the middle one is exactly out of phase with amplitude $\sqrt{2}$ times bigger.

11.27 ★★ Consider two carts of equal mass m on a horizontal, frictionless track. The carts are connected to each other by a single spring of force constant k, but are otherwise free to move freely along the track. **(a)** Write down the Lagrangian and find the normal frequencies of the system. Show that one of the normal frequencies is zero. **(b)** Find and describe the motion in the normal mode whose frequency is nonzero. **(c)** Do the same for the mode with zero frequency. [*Hint:* This one requires some thought. It isn't immediately clear what oscillations of zero frequency are. Notice that the eigenvalue equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$ reduces to $\mathbf{Ka} = 0$ in this case. Consider a solution $\mathbf{x}(t) = \mathbf{a}\,f(t)$, where $f(t)$ is an undetermined function of t and use the equation of motion, $\mathbf{M\ddot{x}} = -\mathbf{Kx}$, to show that this solution represents motion of the whole system with constant velocity. Explain why this kind of motion is possible here but not in the previous examples.]

(a) The Lagrangian is

$$\mathcal{L} = T - U = \tfrac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \tfrac{1}{2}k(x_1 - x_2)^2 = \tfrac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \tfrac{1}{2}k(x_1^2 - 2x_1x_2 + x_2^2)$$

so the matrices **M** and **K** are

$$\mathbf{M} = m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{K} = m\omega_o^2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(where $\omega_o^2 = k/m$) and $\det(\mathbf{K} - \omega^2\mathbf{M}) = m^2\omega^2(\omega^2 - 2\omega_o^2)$. Therefore, the two normal frequencies are $\omega_1 = 0$ and $\omega_2 = \sqrt{2}\omega_o$.

(b) If we put $\omega = \omega_2$, the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$ requires that $a_2 = -a_1$. That is, in mode 2, the two carts oscillate exactly out of phase while their CM remains stationary.

(c) If we put $\omega = \omega_1 = 0$, the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$ reduces to $\mathbf{Ka} = 0$, which requires that $a_2 = a_1$; that is, $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (times any constant). If we try a solution of the form $\mathbf{x}(t) = \mathbf{a}\,f(t)$, then the equation of motion $\mathbf{M\ddot{x}} = -\mathbf{Kx}$ becomes $\ddot{f} = 0$, so that $f(t) = x_o + v_o t$. In this mode the separation of the two carts is constant and their CM moves with constant velocity. The mode is possible because neither cart is attached to any fixed point, so there are no external forces; total momentum is conserved, and one possible motion is uniform motion of the CM with no internal motion.

11.29 ★★★ A thin rod of length $2b$ and mass m is suspended by its two ends with two identical vertical springs (force constant k) that are attached to the horizontal ceiling. Assuming that the whole system is constrained to move in just the one vertical plane, find the normal frequencies and normal modes of small oscillations. Describe and explain the normal modes. [*Hint:* It is crucial to make a wise choice of generalized coordinates. One possibility would be r, ϕ, and α, where r and ϕ specify the position of the rod's CM relative to an origin half way between the springs on the ceiling, and α is the angle of tilt of the rod. Be careful when writing down the potential energy.]

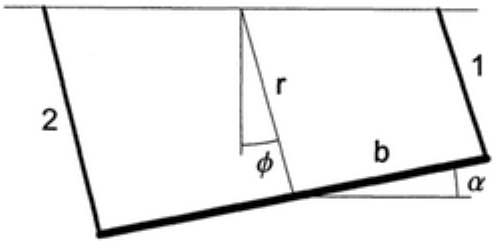

Figure S11.29

I'll introduce the following notations:

L_o = unstretched length of either spring

r_o = distance of rod below ceiling in equilibrium

(r, ϕ) = position of CM of rod

α = angle between rod and horizontal

For small oscillations:

L_1 = (length of spring 1) $\approx r - b\alpha$

L_2 = (length of spring 2) $\approx r + b\alpha$

$r = r_o + \epsilon$, where ϵ is small.

In equilibrium, the total tension in the two springs must balance the weight of the rod, so

$$2k(r_o - L_o) = mg. \tag{S11.11}$$

Spring 1 is stretched by the amount $L_1 - L_o$, so its PE is $U_1 = \frac{1}{2}k(L_1 - L_o)^2 = \frac{1}{2}k(r - L_o - b\alpha)^2$. Similarly, spring 2 has PE $U_2 = \frac{1}{2}k(L_2 - L_o)^2 = \frac{1}{2}k(r - L_o + b\alpha)^2$. Therefore, the total spring PE is

$$\begin{aligned} U_{sp} = U_1 + U_2 &= k(r - L_o)^2 + kb^2\alpha^2 \\ &= k(r_o - L_o + \epsilon)^2 + kb^2\alpha^2 \\ &= \text{const} + 2k(r_o - L_o)\epsilon + k\epsilon^2 + kb^2\alpha^2 \end{aligned} \tag{S11.12}$$

Meanwhile the gravitational PE is

$$U_{gr} = -mgr\cos\phi \approx -mg(r_o + \epsilon)(1 - \phi^2/2) \approx \text{const} - mg\epsilon + mgr_o\phi^2/2 \tag{S11.13}$$

where in the first approximation I set $\cos\phi \approx 1 - \phi^2/2$ and in the second I dropped all terms with products of three or more small quantities. The total PE is found by adding Eqs. (S11.12) and (S11.13). By Eq. (S11.11), the terms linear in ϵ cancel, and we find

$$U = U_{sp} + U_{gr} = k\epsilon^2 + \tfrac{1}{2}mgr_o\phi^2 + kb^2\alpha^2.$$

The KE is

$$T = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \tfrac{1}{2}I\dot{\alpha}^2 \approx \tfrac{1}{2}m\dot{\epsilon}^2 + \tfrac{1}{2}mr_o^2\dot{\phi}^2 + \tfrac{1}{6}mb^2\dot{\alpha}^2.$$

The Lagrangian is $\mathcal{L} = T - U$, as usual.

The r equation is $m\ddot{\epsilon} = -2k\epsilon$. This says the rod can oscillate vertically up and down, with angular frequency $\omega_1 = \sqrt{2k/m}$. The factor of 2 is because both springs are being stretched or compressed.

The ϕ equation is $mr_o^2\ddot{\phi} = -mgr_o\phi$. This says the rod can swing sideways like a simple pendulum with angular frequency $\omega_2 = \sqrt{g/r_o}$.

The α equation is $\frac{1}{3}mb^2\ddot{\alpha} = -2kb^2\alpha$. This says the rod can oscillate (up on the left and down on the right, then vice versa), with angular frequency $\omega_3 = \sqrt{6k/m}$. Notice that because $\mathcal{L}$ involves no cross terms (no terms involving more than one variable) the three Lagrange equations are uncoupled. That is, the three coordinates r, ϕ and α are already normal coordinates and oscillate independently.

11.31 ⋆⋆⋆ Consider a frictionless rigid horizontal hoop of radius R. Onto this hoop I thread three beads with masses $2m$, m, and m, and, between the beads, three identical springs, each with force constant k. Solve for the three normal frequencies and find and describe the three normal modes.

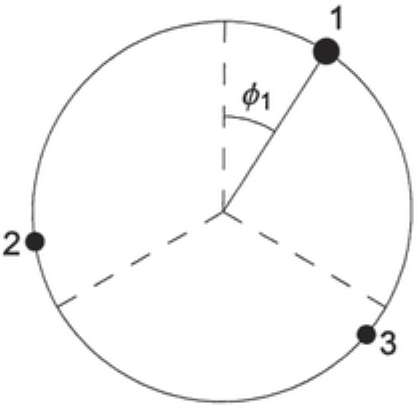

Figure S11.31

The three equilibrium positions are located at 120° to one another, as indicated by the dashed lines. The three masses are $m_1 = 2m$ and $m_2 = m_3 = m$, and their positions are their angles ϕ_1 and so on from their equilibrium positions. The total KE is

$$T = \tfrac{1}{2} m R^2 (2\dot{\phi}_1^2 + \dot{\phi}_2^2 + \dot{\phi}_3^2)$$

and the PE is

$$U = \tfrac{1}{2} k R^2 \left[(\phi_1 - \phi_2)^2 + (\phi_2 - \phi_3)^2 + (\phi_3 - \phi_1)^2 \right]$$

To simplify the writing, I'll use units with $m = k = R = 1$. This means, in particular, that the unit of frequency is $\omega_o = \sqrt{k/m}$, the frequency of a single mass m on a single spring. The matrices **M** and **K** are

$$\mathbf{M} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{K} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

and

$$\mathbf{K} - \omega^2 \mathbf{M} = \begin{bmatrix} 2(1-\omega^2) & -1 & -1 \\ -1 & 2-\omega^2 & -1 \\ -1 & -1 & 2-\omega^2 \end{bmatrix}.$$

Therefore, as you can check, $\det(\mathbf{K} - \omega^2\mathbf{M}) = -2\omega^2(\omega^2 - 2)(\omega^2 - 3)$ and the normal frequencies are $\omega_1 = 0$, $\omega_2 = \sqrt{2}$, and $\omega_3 = \sqrt{3}$. (In arbitrary units the last two would be $\omega_2 = \sqrt{2}\omega_o$, and $\omega_3 = \sqrt{3}\omega_o$.) If we put $\omega = 0$ in the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$, we find that $a_1 = a_2 = a_3$. Thus, in the first mode the three masses move with constant speed around the hoop at their equilibrium separation. (Compare Problems 11.27 or 11.28.) If we put $\omega = \omega_2$, we find $a_1 = -a_2 = -a_3$; thus, in the second mode, mass 1 oscillates one way while the other two oscillate the other. Finally, with $\omega = \omega_3$, we find $a_1 = 0$ and $a_2 = -a_3$, and in the third mode, mass 1 is stationary, while 2 and 3 oscillate 180° out of phase.

SECTION 11.7 Normal Coordinates

11.33★ The eigenvectors $\mathbf{a}_{(1)}$ and $\mathbf{a}_{(2)}$ that describe the motion in the two normal modes of the two carts of Section 11.2 are given in (11.78). Prove that *any* (2×1) column $\mathbf{x}$ can be written as a linear combination of these two eigenvectors; that is, $\mathbf{a}_{(1)}$ and $\mathbf{a}_{(2)}$ are a *basis* of the space of (2×1) columns.

From (11.78) $\mathbf{a}_{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{a}_{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. If $\mathbf{r}$ is any 2×1 column, $\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}$, then by inspection

$$\mathbf{r} = \frac{x+y}{2}\mathbf{a}_{(1)} + \frac{x-y}{2}\mathbf{a}_{(2)}$$

11.35★★ Consider the two coupled pendulums of Problem 11.14. **(a)** What would be a natural choice for the normal coordinates ξ_1 and ξ_2? **(b)** Show that even if both pendulums are subject to a resisitive force of magnitude bv (with b small), the equations of motion for ξ_1 and ξ_2 are still uncoupled. **(c)** Find and describe the motion of the two pendulums for the two modes.

(a) The first normal mode has $\phi_1 = \phi_2$ and the second has $\phi_1 = -\phi_2$. Thus if we define $\xi_1 = \phi_1 + \phi_2$ and $\xi_2 = \phi_1 - \phi_2$, the first mode is an oscillation of ξ_1 with ξ_2 remaining zero, and the second mode is the other way round.

(b) The gravitational torque on either pendulum has the form $\Gamma_{\text{grav}} = -mgL\sin\phi \approx -mgL\phi$. The tension in the spring is approximately $kL(\phi_2 - \phi_1)$ and the velocity of either bob has the form $v = L\dot{\phi}$. The equations of motion in the form $I\ddot{\phi} = \Gamma$ read (after a little tidying up)

$$\ddot{\phi}_1 = -\left(\frac{g}{L} + \frac{k}{m}\right)\phi_1 + \frac{k}{m}\phi_2 - 2\beta\dot{\phi}_1 \qquad \text{and} \qquad \ddot{\phi}_2 = \frac{k}{m}\phi_1 - \left(\frac{g}{L} + \frac{k}{m}\right)\phi_2 - 2\beta\dot{\phi}_2$$

where $2\beta = b/m$. Adding these equations gives an equation for ξ_1 and subtracting them gives a separate equation for ξ_2:

$$\ddot{\xi}_1 = -\frac{g}{L}\xi_1 - 2\beta\dot{\xi}_1 \qquad \text{and} \qquad \ddot{\xi}_2 = -\left(\frac{g}{L} + \frac{2k}{m}\right)\xi_2 - 2\beta\dot{\xi}_2.$$

(c) These equations can be solved as in Section 5.4 to give

$$\xi_1 = A_1 e^{-\beta t}\cos(\omega_1 t - \delta_1) \qquad \text{and} \qquad \xi_2 = A_2 e^{-\beta t}\cos(\omega_2 t - \delta_2)$$

where $\omega_1 = \sqrt{(g/L) - \beta^2}$ and $\omega_2 = \sqrt{(g/L) + (2k/m) - \beta^2}$, and A_1, δ_1, A_2, δ_2 are constants determined by the initial conditions. Both modes die out exponentially with the same decay constant β. In mode 1 ($\xi_2 = 0$) the two pendulums oscillate with frequency ω_1. In mode 2 ($\xi_1 = 0$), they oscillate out of phase with frequency ω_2.

PART II

FURTHER TOPICS

CHAPTER 12

Nonlinear Mechanics and Chaos

PROBLEMS AND SOLUTIONS

Warning: Even when the motion is nonchaotic, it can be very sensitive to tiny errors. In several of the computer problems you may need to increase your working precision to get satisfactory results.

SECTION 12.1 Linearity and Nonlinearity

12.1 ⋆ Consider the nonlinear first-order equation $\dot{x} = 2\sqrt{x-1}$. **(a)** By separating variables, find a solution $x_1(t)$. **(b)** Your solution should contain one constant of integration k, so you might reasonably expect it to be the general solution. Show, however, that there is another solution, $x_2(t) = 1$, that is not of the form of $x_1(t)$ whatever the value of k. **(c)** Show that although $x_1(t)$ and $x_2(t)$ are solutions, neither $Ax_1(t)$, nor $Bx_2(t)$, nor $x_1(t) + x_2(t)$ are solutions. (That is, the superposition principle does not apply to this equation.)

(a) The equation $\dot{x} = 2\sqrt{x-1}$ can be separated as $dx/(2\sqrt{x-1}) = dt$, which can be integrated to give $\sqrt{x-1} = t + k$. Solving for x we see that the function $x_1(t) = (t+k)^2 + 1$ is a solution of the original equation for any value of the constant k.
(b) Clearly the function $x_2(t) = 1$ satisfies the original equation and, equally clearly, it is not of the form $x_1(t)$ for any choice of the constant k.
(c) If $x_3 = Ax_1$, then $\dot{x}_3 = A\dot{x}_1 = 2A\sqrt{x_1 - 1} = 2A\sqrt{x_3/A - 1}$ which is not equal to $2\sqrt{x_3 - 1}$ unless $A = 1$. Similarly, if $x_4 = Bx_2 = B$, then $\dot{x}_4 = 0$, which is not equal to $2\sqrt{x_4 - 1}$ unless $B = 1$. Finally, if $x_5 = x_1 + x_2 = x_1 + 1$, then $\dot{x}_5 = \dot{x}_1 = 2\sqrt{x_1 - 1} = 2\sqrt{x_5 - 2}$, which is not equal to $2\sqrt{x_5 - 1}$.

12.3 ⋆ Consider a second-order linear homogeneous equation of the form (12.6). **(a)** Write out a detailed proof of the superposition principle, that if $x_1(t)$ and $x_2(t)$ are solutions of this equation, then so is any linear combination $a_1x_1(t) + a_2x_2(t)$, where a_1 and a_2 are any two constants. **(b)** Consider now the nonlinear equation in which the third term of (12.6) is replaced by $r(t)\sqrt{x(t)}$. Explain clearly why the superposition principle does not hold for this equation.

(a) If x_1 and x_2 are solutions of (12.6), then

$$p\ddot{x}_1 + q\dot{x}_1 + rx_1 = 0 \qquad \text{and} \qquad p\ddot{x}_2 + q\dot{x}_2 + rx_2 = 0.$$

If we multiply the first of these by any constant a_1 and the second by a_2 and add, we find

$$p(a_1\ddot{x}_1 + a_2\ddot{x}_2) + q(a_1\dot{x}_1 + a_2\dot{x}_2) + r(a_1x_1 + a_2x_2) = 0.$$

That is, the function $a_1x_1 + a_2x_2$ is also a solution of the same differential equation.
(b) If x_1 and x_2 are solutions of the proposed nonlinear equation,

$$p\ddot{x} + q\dot{x} + r\sqrt{x} = 0 \tag{S12.1}$$

then

$$p\ddot{x}_1 + q\dot{x}_1 + r\sqrt{x_1} = 0 \qquad \text{and} \qquad p\ddot{x}_2 + q\dot{x}_2 + r\sqrt{x_2} = 0. \tag{S12.2}$$

Let us consider whether the function $x = x_1 + x_2$ is also a solution of this same equation. To this end we try adding the two equations (S12.2) to give

$$p\ddot{x} + q\dot{x} + r(\sqrt{x_1} + \sqrt{x_2}) = 0. \tag{S12.3}$$

This equation is nearly, but not quite, the proposed equation (S12.1). Unfortunately, unless either x_1 or x_2 is zero, $\sqrt{x_1} + \sqrt{x_2} \neq \sqrt{x_1 + x_2}$. Thus because $x = x_1 + x_2$ satisfies (S12.3), it does *not* satisfy (S12.1), and the superposition principle does not hold for the nonlinear equation (S12.1). [Another, slightly simpler, way to show the same thing would be to prove that ax_1 does not satisfy (S12.1) unless $a = 0$ or 1.]

SECTION 12.3 Some Expected Features of the DDP

12.5 ⋆ Use Euler's relation and the corresponding expression for $\cos\phi$ (inside the front cover) to prove the identity (12.15).

Because $\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$,

$$\begin{aligned}\cos^3 x &= \tfrac{1}{8}\left(e^{3ix} + 3e^{ix} + 3e^{-ix} + e^{-3ix}\right)\\ &= \tfrac{1}{4}\left[\tfrac{1}{2}(e^{3ix} + e^{-3ix}) + \tfrac{3}{2}(e^{ix} + e^{-ix})\right] = \tfrac{1}{4}(\cos 3x + 3\cos x).\end{aligned}$$

12.6 ⋆⋆ [Computer] **(a)** Use appropriate sofware to solve the equation (12.11) numerically, for a DDP with the following parameters: drive strength $\gamma = 0.9$, drive frequency $\omega = 2\pi$, natural frequency $\omega_o = 1.5\omega$, damping constant $\beta = \omega_o/4$, and initial conditions $\phi(0) = \dot{\phi}(0) = 0$. Solve the equation and plot your solution for six cycles, $0 \le t \le 6$, and verify that you get the result shown in Figure 12.3. **(b)** and **(c)** Solve the same equation twice more with the two different initial conditions $\phi(0) = \pm\pi/2$ — both with $\dot{\phi}(0) = 0$ — and plot all three solutions on the same picture. Do your results bear out the claim that, for this drive strength, all solutions (whatever their initial conditions) approach the same periodic attractor?

SECTION 12.4 The DDP: Approach to Chaos

12.7 ⋆⋆ [Computer] Do all the same calculations as in Problem 12.6 but with a drive strength $\gamma = 1.06$ and for $0 \le t \le 10$. In part (a) verify that your results agree with Figure 12.4. Do your results suggest that there is still a unique attractor to which all solutions (whatever their initial conditions) converge? (At first sight the answer may appear to be "No," but remember that values of ϕ that differ by 2π should be considered to be the same.)

The solutions with the initial conditions (b) and (c) merge within about 3 cycles and oscillate nearly sinusoidally about $\theta = 0$. The solution (a) is much slower to settle down and eventually oscillates about $\theta = 2\pi$. To check that all three solutions do really have the same long-term behavior, you could plot $\theta(t) - 2\pi$ for solution (a); when this is done, all three agree perfectly (on the scale shown) after about $t = 8$.

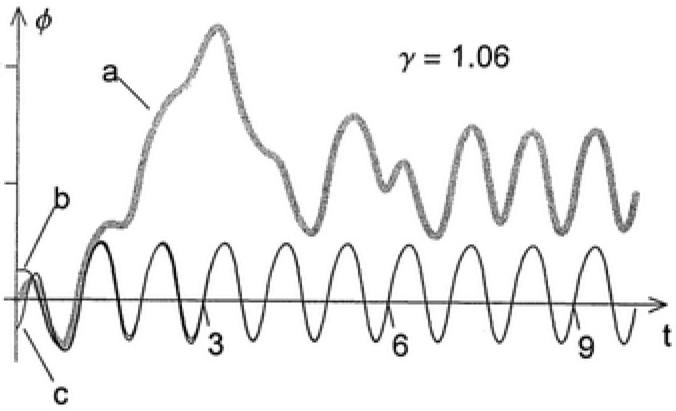

Figure S12.7

12.8 ★★ [Computer] Use a computer to find a numerical solution of the equation of motion (12.11) for a DDP with the following parameters: drive strength $\gamma = 1.073$, drive frequency $\omega = 2\pi$, natural frequency $\omega_o = 1.5\omega$, damping constant $\beta = \omega_o/4$, and initial conditions $\phi(0) = \pi/2$ and $\dot{\phi}(0) = 0$. (**a**) Solve for $0 \le t \le 50$, and then plot the first ten cycles, $0 \le t \le 10$. (**b**) To be sure that the initial transients have died out, plot the ten cycles $40 \le t \le 50$. What is the period of the long-term motion (the attractor)?

12.9 ★★ [Computer] Do the same calculations as in Problem 12.8 with all the same parameters except that $\phi(0) = 0$. Plot the first 30 cycles $0 \le t \le 30$ and check that you agree with Figure 12.5. Plot the ten cycles $40 \le t \le 50$ and find the period of the long-term motion.

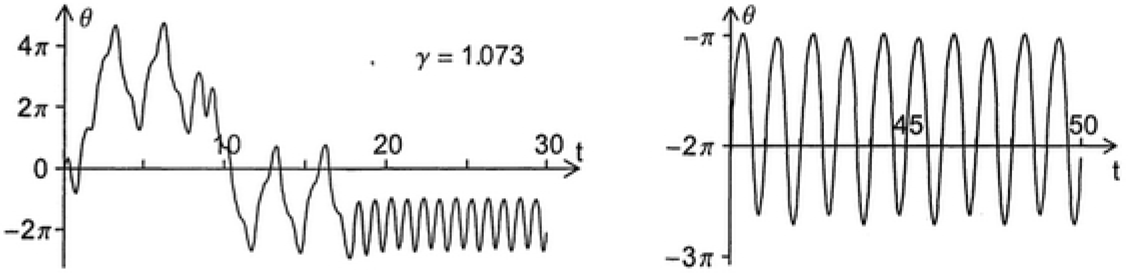

Figure S12.9

The left plot agrees well with Figure 12.5(a). From the right plot it is pretty clear that the period is 2. [Important note on precision: Although chaos has not set in, the solution is quite sensitive to initial conditions and hence also to rounding errors. I found the curves shown using Mathematica's NDSolve function, starting with the default precision of 15 significant figures. I then increased the precision to 16 (WorkingPrecision → 16), and again to 17, and on to 21 in integer steps. The curves for precision 15 and 16 looked quite like one another, but when I increased the precision to 17 the curve changed radically. From then on it made no visible change, indicating that the rounding errors had probably become negligible (for the range $0 \le t \le 50$)

12.11 ★★ Test how well the values of the thresholds γ_n given in Table 12.1 fit the Feigenbaum relation (12.17) as follows: (**a**) Assuming that the Feigenbaum relation is exactly true use it to prove that $(\gamma_{n+1} - \gamma_n) = (1/\delta)^{n-1}(\gamma_2 - \gamma_1)$ and, hence, that a plot of $\ln(\gamma_{n+1} - \gamma_n)$ against n should be a straight line with slope $-\ln\delta$. (**b**) Make this plot for the three differences of Table 12.1. How well do your points seem to bear out our prediction? Fit a line to your plot (either graphically or using a least squares fit) and find the slope and hence the Feigenbaum number δ. [You would not expect to get very good agreement with the known value (12.18) for two reasons: You have only three points to plot and the Feigenbaum relation (12.17) is only approximate, except in the limit of large n. Under the circumstances, you will find the agreement is remarkable.]

(a) $(\gamma_{n+1}-\gamma_n)=\frac{1}{\delta}(\gamma_n-\gamma_{n-1})=\frac{1}{\delta^2}(\gamma_{n-1}-\gamma_{n-2})=\cdots\cdots=\frac{1}{\delta^{n-1}}(\gamma_2-\gamma_1)$. Therefore

$$\ln(\gamma_{n+1}-\gamma_n)=-(n-1)\ln\delta+\ln(\gamma_2-\gamma_1).$$

(b) The least-squares line has slope -1.54, giving $\delta=e^{1.54}=4.66$, compared with the correct value 4.67

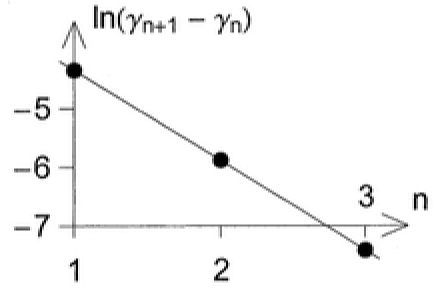

Figure S12.11

SECTION 12.5 Chaos and Sensitivity to Initial Conditions

12.13 ⋆ You can see in Figure 12.13 that for $\gamma=1.105$, the separation of two identical pendulums with slightly different initial conditions increases exponentially. Specifically $\Delta\phi$ starts out at 10^{-4} and by $t=14.5$ it has reached about 1. Use this to estimate the Liapunov exponent λ as defined in (12.26), $\Delta\phi(t)\sim Ke^{\lambda t}$. Your answer should confirm that $\lambda>0$ for chaotic motion.

The crests in Fig.12.13 fall pretty close to a straight line with positive slope, confirming that the crests of $\Delta\phi$ grow exponentially. From the graph we see that $\Delta\phi(0)=10^{-4}$ and $\Delta\phi(14.5)\approx 1$. According to Eq. (12.26), $\Delta\phi(t)/\Delta\phi(0)=e^{\lambda t}$. Therefore

$$\lambda=\frac{1}{t}\ln\left(\frac{\Delta\phi(t)}{\Delta\phi(0)}\right)=\frac{1}{14.5}\ln\left(\frac{1}{10^{-4}}\right)\approx 0.64.$$

12.15 ⋆⋆ [Computer] Numerically solve the equation of motion (12.11) for a DDP with the following parameters: drive strength $\gamma=0.3$, drive frequency $\omega=2\pi$, natural frequency $\omega_o=1.5\omega$, damping constant $\beta=\omega_o/4$, and initial conditions $\phi(0)=\dot\phi(0)=0$. Solve for the first five drive cycles $(0\le t\le 5)$ and call your solution $\phi_1(t)$. Solve again for all the same parameters except that $\phi(0)=1$ (that is, the initial angle is one radian) and call this solution $\phi_2(t)$. Let $\Delta\phi(t)=\phi_2(t)-\phi_1(t)$ and make a plot of $\log|\Delta\phi(t)|$ against t. Does your plot confirm that $\Delta\phi(t)$ goes to zero exponentially? Note: The exponential decay continues indefinitely, but $\Delta\phi(t)$ eventually gets so small that it is smaller than the rounding errors, and the exponential decay cannot be seen. If you want to go further, you will probably need to crank up your precision.

From $t=0$ to 5, the crests of $\log|\Delta\phi(t)|$ decrease perfectly linearly, confirming that $\Delta\phi(t)$ decreases exponentially.

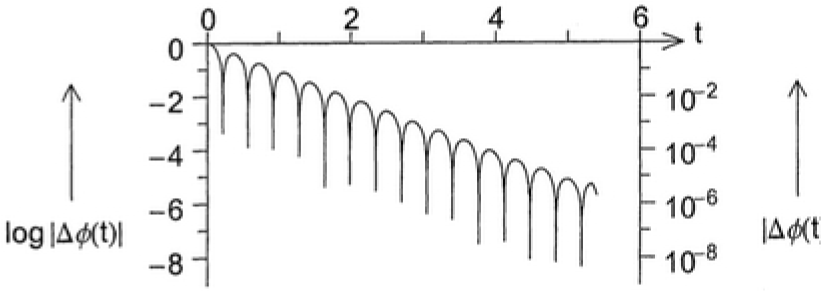

Figure S12.15

12.17 ⋆⋆ [Computer] In Figure 12.15, you can see that for $\gamma = 1.503$ the DDP "tries" to execute a steady rolling motion changing by 2π once each cycle, but that there is superposed an erratic wobbling and that the direction of the rolling reverses itself from time to time. For other values of γ, the pendulum actually does approach a steady, periodic rolling. **(a)** Solve the equation of motion (12.11) for a drive strength $\gamma = 1.3$ and all other parameters as in the first part of Problem 12.14, for $0 \le t \le 8$. Call your solution $\phi_1(t)$ and plot it as a function of t. Describe the motion. **(b)** It is hard to be sure that the motion is periodic based on this graph, because of the steady rolling through -2π each cycle. As a better check, plot $\phi_1(t) + 2\pi t$ against time. Describe what this shows. This kind of periodic rolling motion is sometimes described as **phase-locked**.

(a) The pendulum rolls through -2π each cycle, with a regular looking oscillation superposed.

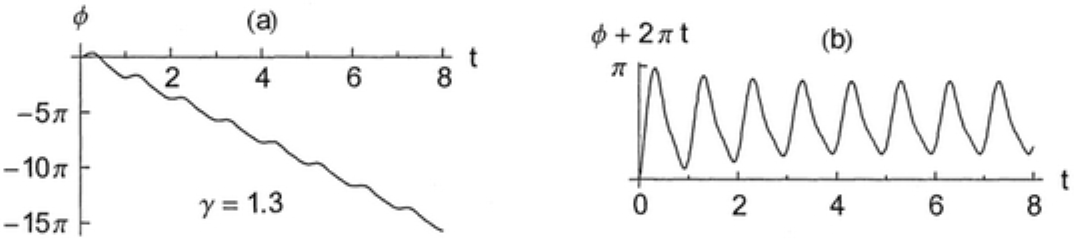

Figure S12.17

(b) In the plot of $\phi(t) + 2\pi t$, with the rolling motion subtracted out, we can see clearly that the oscillations become very regular after two or three cycles.

SECTION 12.7 State-Space Orbits

12.19 ⋆ Consider an undamped, undriven simple harmonic oscillator — a mass m on the end of a spring whose force constant is k. **(a)** Write down the general solution $x(t)$ for the position as a function of time t. Use this to sketch the state-space orbit, showing the motion of the point $[x(t), \dot{x}(t)]$ in the two-dimensional state space with coordinates $(x, \dot{x})$. Explain the direction in which the orbit is traced as time advances. **(b)** Write down the total energy of the system and use conservation of energy to prove that the state-space orbit is an ellipse.

(a) The general solution is $x = A\cos(\omega t - \delta)$ and hence $\dot{x} = -\omega A \sin(\omega t - \delta)$, where $\omega = \sqrt{k/m}$, and A and δ are arbitrary constants. These are the parametric equations of an ellipse in the plane of $(x, \dot{x})$. In the upper half plane ($\dot{x} > 0$), x is increasing and the orbit moves to the right; in the lower half plane ($\dot{x} < 0$), x is decreasing and the orbit moves to the left. Therefore, the orbit is traced clockwise.

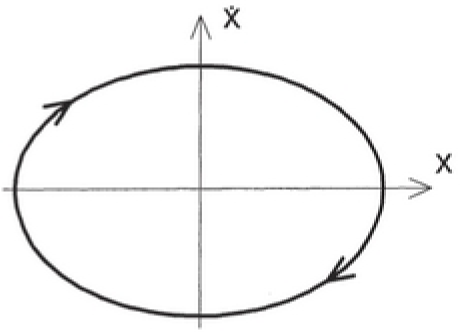

Figure S12.19

(b) $E = \frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2$ = const, which is the equation of an ellipse.

SECTION 12.9 The Logistic Map

12.21 ★★ Here is an iterated map that is easily studied with the help of your calculator: Let $x_{t+1} = f(x_t)$ where $f(x) = \cos(x)$. If you choose any value for x_0, you can find $x_1, x_2, x_3, \cdots$ by simply pressing the cosine button on your calculator over and over again. (Be sure the calculator is in radians mode.) **(a)** Try this for several different choices of x_0, finding the first 30 or so values of x_t. Describe what happens. **(b)** You should have found that there seems to be a single fixed attractor. What is it? Explain it, by examining (graphically, for instance) the equation for a fixed point $f(x^*) = x^*$ and applying our test for stability [namely, that a fixed point x^* is stable if $|f'(x^*)| < 1$].

(a) Here are the values of x_t for three different initial values, $x_0 = 0$, 3, and 100.

x_0	x_1	x_2	x_3	x_4	x_5	$\cdots$	x_{28}	x_{29}	x_{30}
0	1.00	0.54	0.86	0.65	0.793	$\cdots$	0.7391	0.7391	0.7391
3	−0.99	0.55	0.85	0.66	0.791	$\cdots$	0.7391	0.7391	0.7391
100	0.86	0.65	0.80	0.70	0.765	$\cdots$	0.7391	0.7391	0.7391

For all three initial values x_0, the sequence converges on the attractor $x = 0.7391$.
(b) The attractor must be a solution of the fixed-point equation $f(x^*) = x^*$. In the present case, this is $\cos(x^*) = x^*$, which has exactly one solution, $x^* = 0.739085$, and this is stable since $|f'(x^*)| < 1$.

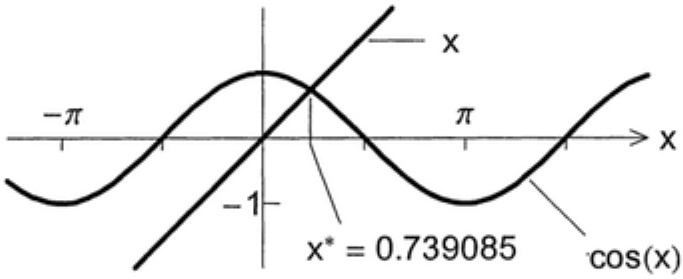

Figure S12.21

12.23 ★★ [Computer] Consider the **sine map** $x_{t+1} = f(x_t)$ where $f(x) = r\sin(\pi x)$. The interesting behavior of this map is for $0 \le x \le 1$ and $0 \le r \le 1$, so restrict your attention to these ranges. **(a)** Using a plot analogous to Figure 12.35, discuss the fixed points of this map. Show that the map has either one or two fixed points, depending on the value of r. Show that when r is small there is just one fixed point, which is stable. **(b)** At what value of r (call it r_0) does the second fixed point appear? Show that r_0 is also the value of r at which the first fixed point becomes unstable. **(c)** As r increases, the second fixed point eventually becomes unstable. Find numerically the value r_1 at which this occurs.

(a) and **(b)** The fixed points x^* must satisfy $x^* = f(x^*)$; that is, they are determined by the intersections (if any) of the line $y = x$ and the curve $y = f(x)$, where, in this case, $f(x) = r \sin \pi x$. As you can see from the picture, there is always a fixed point at $x^* = 0$. Because the slope of the curve $y = f(x)$ at $x = 0$ is $f'(0) = r\pi$, we see that when $r < 1/\pi$, the only fixed point is at $x^* = 0$, but if we increase r until $r > 1/\pi$, then a second fixed point appears. The value of r at which this happens is evidently $r_0 = 1/\pi$.

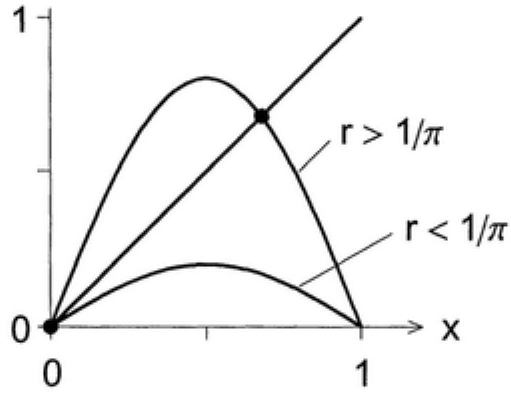

Figure S12.23

The test for stability is that $|f(x^*)| < 1$. Thus the fixed point at $x^* = 0$ is stable provided $f'(0) = r\pi < 1$ and unstable if $r\pi > 1$. Therefore this first fixed point becomes unstable at the same value $r_0 = 1/\pi$ at which the second fixed point appears.

(c) The second fixed point at $x^* > 0$ is stable as long as $|f'(x^*)| < 1$. As we increase r, this fixed point becomes unstable at the value $r = r_1$ for which $f'(x^*) = -1$. This transition is determined by the two equations

$$f(x^*) = x^* \quad \text{that is,} \quad r_1 \sin(\pi x^*) = x^*$$

and

$$f'(x^*) = -1 \quad \text{that is,} \quad r_1 \pi \cos(\pi x^*) = -1.$$

These two simultaneous equations for r_1 and x^* can be solved numerically in several ways. (One straightforward method is to use Mathematica's FindRoot.) The result is that $r_1 = 0.71996$.

12.25 ⋆⋆ [Computer] The sine map of Problem 12.23 exhibits period-doubling cascades just like the logistic map. To illustrate this, take $x_0 = 0.8$ and find the first twenty values x_t for each of the following values of the parameter r: **(a)** $r = 0.60$, **(b)** $r = 0.79$, **(c)** $r = 0.85$, and **(d)** $r = 0.865$. Plot your results (as four separate plots) and comment.

When $r = 0.6$, x_t approaches a single attractor at $x^* = 0.58$. When $r = 0.79$, x_t is exhibiting period 2 oscillations, bouncing between 0.49 and 0.79. By the time $r = 0.85$ the period has doubled again to period 4, and when $r = 0.865$ it is period 8.

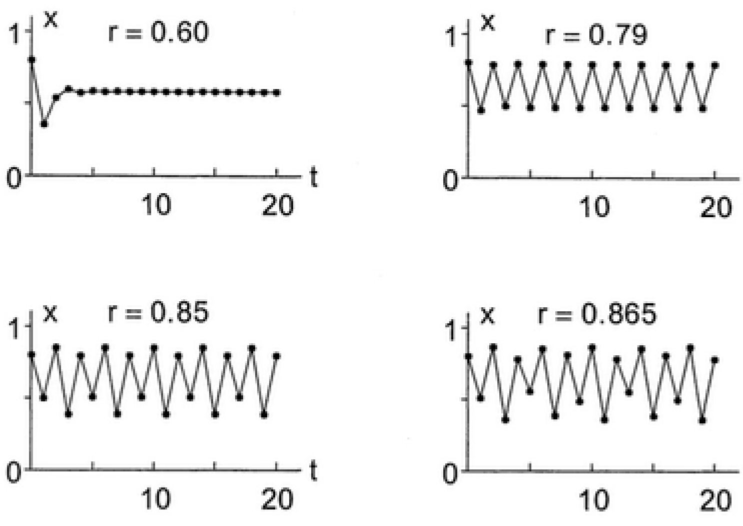

Figure S12.25

12.27 ★★ The two-cycles of the map $f(x)$ correspond to fixed points of the second-iterate $g(x) = f(f(x))$. Thus the two values shown as x_a and x_b in Figure 12.37 are roots of the equation $f(f(x)) = x$. In the case of the logistic map this is a quartic equation, which is not too hard to solve: **(a)** Verify that for the logistic function $f(x)$

$$x - f(f(x)) = rx\left(x - \frac{r-1}{r}\right)\Big[r^2x^2 - r(r+1)x + r + 1\Big]. \tag{12.61}$$

Thus the fixed-point equation $x - f(f(x)) = 0$ has four roots. The first two are $x = 0$ and $x = (r-1)/r$. Explain these and show that the other two roots are

$$x_a, x_b = \frac{r + 1 \pm \sqrt{(r+1)(r-3)}}{2r}. \tag{12.62}$$

Explain how you know that these are the two points of a two-cycle. **(b)** Show that for $r < 3$ these roots are complex and hence that there is no real two-cycle. **(c)** For $r \geq 3$ these two roots are real and there is a real two-cycle. Find the values of x_a and x_b for the case $r = 3.2$ and verify the values shown in Figure 12.37.

(a) Because $f(x) = rx(1-x)$

$$f[f(x)] = rf(x)[1 - f(x)] = r^2x(1-x)[1 - rx(1-x)] = -r^2x[rx^3 - 2rx^2 + (1+r)x - 1].$$

Therefore

$$\begin{aligned} x - f[f(x)] &= x[r^3x^3 - 2r^3x^2 + r^2(r+1)x + (1-r^2)] \\ &= x(rx - r + 1)[r^2x^2 - r(r+1)x + r + 1] \end{aligned}$$

in agreement with (12.61).

The fixed points of $f[f(x)]$ are the zeroes of this expression. The first two are $x = 0$ and $x = (r-1)/r$, which are just the two known fixed points of $f(x)$. The other two are the zeroes of the quadratic expression in square brackets and are

$$x_a, x_b = \frac{r(r+1) \pm \sqrt{r^2(r+1)^2 - 4r^2(r+1)}}{2r^2} = \frac{(r+1) \pm \sqrt{(r+1)(r-3)}}{2r}$$

in agreement with (12.62). Because these are fixed points of $f[f(x)]$ but not of $f(x)$, they have to be the two points of a two-cycle of $f(x)$.

(b) It is always true that $0 \le r \le 4$. If, in addition, $r < 3$, the product inside the square root is negative and both x_a and x_b are complex. Therefore there are no real two-cycles for $r < 3$.

(c) Putting $r = 3.2$ in our expressions for x_a and x_b, we get the two values 0.51 and 0.80, in apparent agreement with Fig.12.37.

12.29 ⋆⋆ [Computer] The thresholds r_n for period doubling of the logistic map are given by Equation (12.58). These should satisfy the Feigenbaum relation (12.17), at least in the limit that $n \to \infty$ (with γ replaced by r, of course). Test this claim as follows: **(a)** If you have not done Problem 12.11 prove that the Feigenbaum relation (if exactly true) implies that $(r_{n+1} - r_n) = K/\delta^n$. **(b)** Make a plot of $\ln(r_{n+1} - r_n)$ against n. Find the best-fit straight line to the data and from its slope predict the Feigenbaum constant. How does your answer compare with the accepted value $\delta = 4.67$?

(a) See solution to Problem 12.11.

(b) From the values of r_n in Eq. (12.58) we can calculate the following:

n	1	2	3	4	5
$\ln(r_{n+1} - r_n)$	−0.800	−2.358	−3.897	−5.426	−7.013

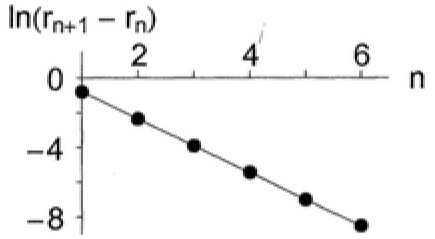

Figure S12.29

A least-squares fit to the line $\ln(r_{n+1} - r_n) = \ln(K) - n\ln(\delta)$ yields $\ln(K) = 0.7396$ and $\ln(\delta) = 1.5452$, whence $\delta = 4.69$, compared with the accepted value $\delta = 4.67$. The data and the best-fit line are plotted above.

12.31 ⋆⋆⋆ [Computer] When the evolution of the logistic map is non-chaotic, two solutions with the same r that start out sufficiently close, will converge exponentially. (This was illustrated in Problem 12.30.) This does not mean that *any* two solutions with the same r will converge. **(a)** Repeat Problem 12.30(a), with all the same parameters except $r = 3.5$ (a value for which we know the long-term motion has period 4). Does $x'_t - x_t$ approach zero? **(b)** Now do the same exercise but with the initial conditions $x_0 = 0.45$ and $x'_0 = 0.5$. Comment. Can you explain why, if the period is greater than 1, it is impossible that the difference $x'_t - x_t$ go to zero for all choices of initial conditions?

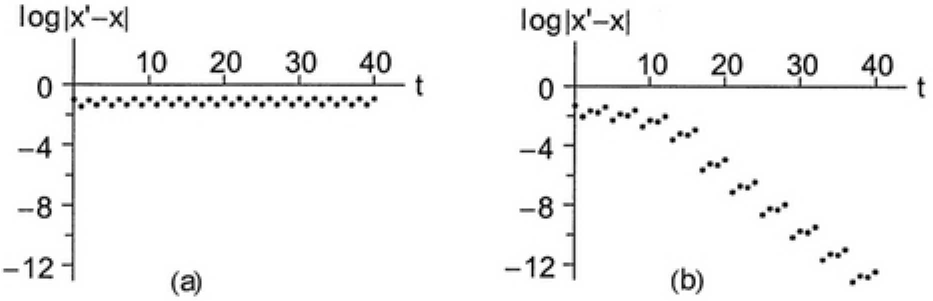

Figure S12.31

(a) With $x_0 = 0.4$ and $x'_0 = 0.5$, the difference $x'_t - x_t$ remains pretty constant with a small periodic oscillation.

(b) Here with $x_0 = 0.45$ and $x'_0 = 0.5$, the difference $|x'_t - x_t| \to 0$ exponentially.

To understand these different behaviors, consider the case of a period-2 orbit x_t with fixed points x_a and x_b. We could launch a second orbit x'_t so that it was exactly one period out of step with the first. (Just take $x'_1 = x_0$.) Then every time the first orbit is at x_a, the second is at x_b, and vice versa. Thus $|x'_t - x_t|$ will be a constant. On the other hand, if we launch the two orbits sufficiently close, they will eventually get into step and $|x'_t - x_t|$ will approach zero.

12.33 ⋆⋆⋆ [Computer] Reproduce the logistic bifurcation diagram of Figure 12.41 for the range $2.8 \le r \le 4$. Take $x_0 = 0.1$. [*Hint:* To make Figure 12.41 I used about 50,000 points, but you certainly don't need to use that many. In any case, start by using a very small number of points, perhaps just r going from 2.8 to 3.4 in steps of 0.2 and t going from 51 to 54. This will let you calculate for each of the values of r individually, and get the feel of how things work. To make a good diagram, you will then need to increase the number of points (r going from 2.8 to 4 in steps of 0.025, and t from 500 to 600, perhaps), and you will certainly need to automate the calculation of the large number of points.]

See Fig.12.41.

CHAPTER 13

Hamiltonian Mechanics

PROBLEMS AND SOLUTIONS

SECTION 13.2 Hamilton's Equations for One-Dimensional Systems

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13.1 ⋆ Find the Lagrangian, the generalized momentum, and the Hamiltonian for a free particle (no forces at all) confined to move along the x axis. (Use x as your generalized coordinate.) Find and solve Hamilton's equations.

$\mathcal{L} = \frac{1}{2}m\dot{x}^2$, $p = \partial\mathcal{L}/\partial\dot{x} = m\dot{x}$, and $\mathcal{H} = p\dot{x} - \mathcal{L} = p^2/2m$. The Hamilton equations are

$$\dot{x} = \partial\mathcal{H}/\partial p = p/m \qquad \text{and} \qquad \dot{p} = -\partial\mathcal{H}/\partial x = 0,$$

with solutions $p = p_o = \text{const}$, and $x = x_o + v_o t$, where $v_o = p_o/m$.

13.3 ⋆ Consider the Atwood machine of Figure 13.2, but suppose that the pulley is a uniform disc of mass M and radius R. Using x as your generalized coordinate, write down the Lagrangian, the generalized momentum p, and the Hamiltonian $\mathcal{H} = p\dot{x} - \mathcal{L}$. Find Hamilton's equations and use them to find the acceleration $\ddot{x}$.

The moment of inertia of a uniform disc is $I = \frac{1}{2}MR^2$, and its kinetic energy is $\frac{1}{2}I\omega^2 = \frac{1}{2}I\dot{x}^2/R^2$. Therefore, $\mathcal{L} = \frac{1}{2}(m_1 + m_2 + \frac{1}{2}M)\dot{x}^2 + (m_1 - m_2)gx$, $p = (m_1 + m_2 + \frac{1}{2}M)\dot{x}$, and $\mathcal{H} = p\dot{x} - \mathcal{L} = p^2/2(m_1 + m_2 + \frac{1}{2}M) - (m_1 - m_2)gx$. The Hamilton equations are

$$\dot{x} = \partial\mathcal{H}/\partial p = p/(m_1 + m_2 + \tfrac{1}{2}M) \qquad \text{and} \qquad \dot{p} = -\partial\mathcal{H}/\partial x = (m_1 - m_2)g,$$

and the acceleration is $\ddot{x} = g(m_1 - m_2)/(m_1 + m_2 + \frac{1}{2}M)$.

13.5 ⋆⋆ A bead of mass m is threaded on a frictionless wire that is bent into a helix with cylindrical polar coordinates (ρ, ϕ, z) satisfying $z = c\phi$ and $\rho = R$, with c and R constants. The z axis points vertically up and gravity vertically down. Using ϕ as your generalized coordinate, write down the kinetic and potential energies, and hence the Hamiltonian $\mathcal{H}$ as a function of ϕ and its conjugate momentum p. Write down Hamilton's equations and solve for $\ddot{\phi}$ and hence $\ddot{z}$. Explain your result in terms of Newtonian mechanics and discuss the special case that $R = 0$.

Since $\rho = R$ is fixed and $z = c\phi$, there is just one generalized coordinate, which we can choose to be ϕ. The bead's kinetic energy is $T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{z}^2 + R^2\dot{\phi}^2) = \frac{1}{2}m(c^2 + R^2)\dot{\phi}^2$, the potential energy is $U = mgz = mgc\phi$, and the generalized momentum is $p = \partial T/\partial\dot{\phi} = m(c^2 + R^2)\dot{\phi}$. From these we find the Hamiltonian:

$$\mathcal{H} = T + U = \frac{p^2}{2m(c^2 + R^2)} + mgc\phi,$$

and the two Hamilton equations:

$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m(c^2 + R^2)} \qquad \text{and} \qquad \dot{p} = -\frac{\partial \mathcal{H}}{\partial \phi} = -mgc.$$

Combining the last two, we can find $\ddot{\phi}$ and thence

$$\ddot{z} = c\ddot{\phi} = -g\frac{c^2}{c^2 + R^2} = -g\sin^2\alpha$$

where in the last expression I have introduced tha pitch α of the helix. (The bead rises a height $2\pi c$ in a horizontal run of $2\pi R$, so $\tan\alpha = c/R$ and $\sin\alpha = c/\sqrt{c^2 + R^2}$.)

According to Newton, we could argue that the bead's tangential acceleration is $a_{\text{tang}} = g\sin\alpha$ (the well known acceleration down an incline) and its vertical component is $\ddot{z} = -a_{\text{tang}}\sin\alpha = -g\sin^2\alpha$. If $R = 0$, then $\alpha = 90°$, and $\ddot{z} = -g$, as expected on a vertical wire.

13.7 ★★★ A roller coaster of mass m moves along a frictionless track that lies in the xy plane (x horizontal and y vertically up). The height of the track above the ground is given by $y = h(x)$. **(a)** Using x as your generalized coordinate, write down the Lagrangian, the generalized momentum p, and the Hamiltonian $\mathcal{H} = p\dot{x} - \mathcal{L}$ (as a function of x and p). **(b)** Find Hamilton's equations and show that they agree with what you would get from the Newtonian approach. [*Hint:* You know from Section 4.7 that Newton's second law takes the form $F_{\text{tang}} = m\ddot{s}$, where s is the distance measured along the track. Rewrite this as an equation for $\ddot{x}$ and show that you get the same result from Hamilton's equations.]

(a) The height of the track is $y = h(x)$. Therefore the distance traveled by the car in a small displacement is

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + h'(x)^2}dx. \tag{S13.1}$$

It follows that the car's speed satisfies $v^2 = (1 + h'^2)\dot{x}^2$, so the Lagrangian and generalized momentum are

$$\mathcal{L} = \tfrac{1}{2}m\dot{x}^2[1 + h'(x)^2] - mgh(x) \qquad \text{and} \qquad p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}[1 + h'(x)^2]. \tag{S13.2}$$

The second equation is easily solved for $\dot{x}$ and the Hamiltonian is

$$\mathcal{H} = \dot{x}p - \mathcal{L} = \frac{p^2}{2m(1 + h'^2)} + mgh$$

(b) Hamilton's equations are

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m(1 + h'^2)} \quad \text{and} \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = \frac{p^2 h' h''}{m(1 + h'^2)^2} - mgh' = m(\dot{x}^2 h' h'' - gh') \tag{S13.3}$$

where in the last step I used Eq. (S13.2) to replace p by $p = m\dot{x}(1 + h'^2)$.

Before we do anything with this Hamiltonian result, let us look at the Newtonian prediction,

$$m\ddot{s} = F_{\text{tang}} = -dU/ds = -mgh'/\sqrt{1 + h'^2}, \tag{S13.4}$$

where, in the last step, I wrote $U = mgh$ and I used Eq. (S13.1) to replace ds by $dx\sqrt{1+h'^2}$. To replace $\ddot{s}$ by $\ddot{x}$, note that

$$\ddot{s} = \frac{d}{dt}\frac{ds}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\sqrt{1+h'^2}\right) = \ddot{x}\sqrt{1+h'^2} + \frac{h'h''\dot{x}^2}{\sqrt{1+h'^2}}.$$

Inserting this result into (S13.4) and solving for $\ddot{x}$ we find that, according to Newton,

$$\ddot{x} = -\frac{gh' + h'h''\dot{x}^2}{1+h'^2}. \tag{S13.5}$$

Let's now see that we get the same result from the Hamilton equations (S13.3). From the first of Eqs. (S13.3) we find

$$\ddot{x} = \frac{d}{dt}\dot{x} = \frac{d}{dt}\frac{p}{m(1+h'^2)} = \frac{\dot{p}}{m(1+h'^2)} - \frac{p}{m}\frac{2h'h''\dot{x}}{(1+h'^2)^2}$$

If we use the second Hamilton equation (S13.3) to eliminate $\dot{p}$ and the second of Eqs. (S13.2) to eliminate p, this is easily seen to be exactly the same as the Newtonian result (S13.5).

There is a much simpler way to accomplish the same result, though it may seem a cheat at first sight. Hamilton's equations, like Lagrange's from which we derived them, are true with respect to any choice of generalized coordinates. Therefore we can handle the same problem using as our generalized coordinate s, the distance measured along the track. If we do this, then the Lagrangian is $\mathcal{L} = \frac{1}{2}m\dot{s}^2 - U(s)$ and the generalized momentum is $p = \partial\mathcal{L}/\partial s = m\dot{s}$. Thus the Hamiltonian is $\mathcal{H} = p^2/(2m) + U(s)$ and the second Hamilton equation is $\dot{p} = -\partial\mathcal{H}/\partial s = -dU/ds$ or $m\ddot{s} = -dU/ds$ in agreement with the Newtonian result Eq. (S13.4).

SECTION 13.3 Hamilton's Equations in Several Dimensions

13.9 ⋆ Set up the Hamiltonian and Hamilton's equations for a projectile of mass m, moving in a vertical plane and subject to gravity but no air resistance. Use as your coordinates x measured horizontally and y measured vertically up. Comment on each of the four equations of motion.

The Lagrangian is $\mathcal{L} = T - U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$, and the generalized momentum has components $p_x = \partial\mathcal{L}/\partial\dot{x} = m\dot{x}$ and $p_y = \partial\mathcal{L}/\partial\dot{y} = m\dot{y}$. Therefore, the Hamiltonian is $\mathcal{H} = \mathbf{p}\cdot\dot{\mathbf{r}} - \mathcal{L} = (p_x^2 + p_y^2)/2m + mgy$. The Hamilton equations are

$$\dot{x} = p_x/m, \quad \dot{y} = p_y/m, \quad \dot{p}_x = 0, \qquad \text{and} \qquad \dot{p}_y = -mg.$$

The first two of these simply reproduce the known relations for $\mathbf{p}$ in terms of $\dot{\mathbf{r}}$. The third says that the x component of $\mathbf{p}$ is constant, and the last that p_y changes at the expected rate $-mg$.

13.11 ⋆ The simple form $\mathcal{H} = T + U$ is true only if your generalized coordinates are "natural" (relation betweeen generalized and underlying Cartesian coordinates is independent of time). If the generalized coordinates are not "natural," you must use the definition $\mathcal{H} = \sum p_i\dot{q}_i - \mathcal{L}$. To illustrate this point, consider the following: Two children are playing catch inside a railroad car that is moving with varying speed V along a straight horizontal track. For generalized coordinates you can use the position (x, y, z) of the ball relative to a point fixed in the car, but in setting up the Hamiltonian you must use coordinates in an inertial frame — a frame fixed to the ground. Find the Hamiltonian for the ball and show that it is not equal to $T + U$ (neither as measured in the car, nor as measured in the ground-based frame).

Let's measure x along the tracks (in the direction of travel), y crossways, and z vertically up, all three relative to the car. The ball's velocity relative to the ground is $(V+\dot{x},\ \dot{y},\ \dot{z})$, so the Lagrangian is $\mathcal{L} = T - U = \frac{1}{2}m[(V+\dot{x})^2 + \dot{y}^2 + \dot{z}^2] - mgz$. The generalized momentum has components

$$p_x = \frac{\partial \mathcal{L}}{\partial x} = m(V+\dot{x}), \qquad p_y = m\dot{y}, \quad \text{and} \quad p_z = m\dot{z}$$

(Notice that, perhaps unexpectedly, the generalized momentum is the momentum relative to the ground, not to the moving car.) We can solve for $\dot{x} = (p_x - mV)/m$ and then

$$\mathcal{H} = p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - \mathcal{L} = \frac{\mathbf{p}^2}{2m} - p_xV + mgz.$$

From this, you can derive the expected equation of motion (as you could check), but our point here is to note that $\mathcal{H}$ is not equal to the energy $T+U$ (neither relative to the car nor relative to the ground), because

$$\begin{aligned}(T+U)_{\text{(rel to car)}} &= \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz \\ &= \frac{\mathbf{p}^2}{2m} - p_xV + \tfrac{1}{2}mV^2 + mgz \neq \mathcal{H}.\end{aligned}$$

and

$$(T+U)_{\text{(rel to ground)}} = \frac{\mathbf{p}^2}{2m} + mgz \neq \mathcal{H}.$$

13.13 ⋆⋆ Consider a particle of mass m constrained to move on a frictionless cylinder of radius R, given by the equation $\rho = R$ in cylindrical polar coordinates (ρ, ϕ, z). The mass is subject to just one external force, $\mathbf{F} = -kr\hat{\mathbf{r}}$, where k is a positive constant, r is its distance from the origin, and $\hat{\mathbf{r}}$ is the unit vector pointing away from the origin, as usual. Using z and ϕ as generalized coordinates, find the Hamiltonian $\mathcal{H}$. Write down and solve Hamilton's equations and describe the motion.

The KE is $T = \frac{1}{2}m(R^2\dot{\phi}^2 + \dot{z}^2)$, and the PE is $U = \frac{1}{2}kr^2 = \frac{1}{2}k(R^2 + z^2)$. Thus the two generalized momenta are

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = mR^2\dot{\phi} \qquad \text{and} \qquad p_z = \frac{\partial T}{\partial \dot{z}} = m\dot{z}$$

and the Hamiltonian is

$$\mathcal{H} = T + U = \frac{1}{2m}\left(\frac{p_\phi^2}{R^2} + p_z^2\right) + \frac{1}{2}k(R^2 + z^2).$$

The two Hamilton equations for z are

$$\dot{z} = \frac{\partial \mathcal{H}}{\partial p_z} = \frac{p_z}{m} \qquad \text{and} \qquad \dot{p}_z = -\frac{\partial \mathcal{H}}{\partial z} = -kz.$$

These two combine to give $\ddot{z} = -(k/m)z$, which shows that the motion in the z direction is SHM with frequency $\omega = \sqrt{k/m}$. The two Hamilton equations for ϕ are

$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{mR^2} \qquad \text{and} \qquad \dot{p}_\phi = -\frac{\partial \mathcal{H}}{\partial \phi} = 0.$$

The first of these simply repeats the relation between $\dot{\phi}$ and p_ϕ. The second tells us that p_ϕ (namely, the z component of angular momentum) is constant and hence that the motion around the cylinder proceeds with constant $\dot{\phi}$.

13.15 ⋆⋆ Fill in the details of the derivation of Hamilton's $2n$ equations (13.25) for a system with n degrees of freedom, starting from Equation (13.24). You can parallel the argument that led from (13.14) to (13.15) and (13.16), but you have $2n$ different derivatives to consider and lots of summations from $i = 1$ to n to contend with.

According to Eq. (13.24)

$$\mathcal{H} = \mathcal{H}(\mathbf{q}, \mathbf{p}, t) = \sum_{j=1}^{n} p_j \dot{q}_j(\mathbf{q}, \mathbf{p}, t) - \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p}, t), t). \tag{S13.6}$$

Differentiating with respect to q_i, we find

$$\frac{\partial \mathcal{H}}{\partial q_i} = \sum_j p_j \frac{\partial \dot{q}_j}{\partial q_i} - \left[\frac{\partial \mathcal{L}}{\partial q_i} + \sum_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial q_i} \right] = -\frac{\partial \mathcal{L}}{\partial q_i} = -\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = -\dot{p}_i$$

where, in the second expression, I replaced $\partial \mathcal{L}/\partial \dot{q}_j$ in the second sum by p_j, so that the two sums cancelled, and for the third equality I used Lagrange's equations. Differentiating (S13.6) with respect to p_i we find

$$\frac{\partial \mathcal{H}}{\partial p_i} = \left[\dot{q}_i + \sum_j p_j \frac{\partial \dot{q}_j}{\partial p_i} \right] - \sum_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial p_i} = \dot{q}_i$$

where, in the second sum of the second expression, I replaced $\partial \mathcal{L}/\partial \dot{q}_j$ by p_j, so that the two sums cancel.

13.17 ⋆⋆⋆ Consider the mass confined to the surface of a cone described in Example 13.4 (page 533). We saw that there are solutions for which the mass remains at the fixed height $z = z_o$, with fixed angular velocity $\dot{\phi}_o$ say. **(a)** For any chosen value of p_ϕ, use (13.34) to get an equation that gives the corresponding value of the height z_o. **(b)** Use the equations of motion to show that this motion is stable. That is, show that if the orbit has $z = z_o + \epsilon$ with ϵ small, then ϵ will oscillate about zero. **(c)** Show that the angular frequency of these oscillations is $\omega = \sqrt{3}\dot{\phi}_o \sin\alpha$, where α is the half angle of the cone ($\tan\alpha = c$ where c is the constant in $\rho = cz$). **(d)** Find the angle α for which the frequency of oscillation ω is equal to the orbital angular velocity $\dot{\phi}_o$, and describe the motion for this case.

(a) From (13.32) we see that $\dot{z} = 0$ if and only if $p_z = 0$, and from (13.34) that this implies that

$$\frac{p_\phi^2}{mc^2 z_o^3} = mg \quad \text{or} \quad z_o = \left(\frac{p_\phi^2}{m^2 c^2 g} \right)^{1/3}.$$

(b) If we combine the two equation (13.34) to give $\ddot{z}$ and then put $z = z_o + \epsilon$, we find

$$\ddot{\epsilon} = \ddot{z} = \frac{\dot{p}_z}{m(c^2+1)} = \frac{1}{m(c^2+1)}\left[\frac{p_\phi^2}{mc^2z^3} - mg\right]$$

$$= \frac{1}{m(c^2+1)}\left[\frac{p_\phi^2}{mc^2{z_o}^3}\left(1 - 3\frac{\epsilon}{z_o}\right) - mg\right] = -\frac{3p_\phi^2\epsilon}{m^2c^2(c^2+1){z_o}^4}$$

where in passing to the second line I used the binomial approximation for z^{-3} and where, in the first expression of the second line, the first and last terms cancelled because of the result of part (a). This equation implies that ϵ oscillates in SHM.

(c) The last equation of part (b) implies that the frequency of these oscillations is

$$\omega = \frac{p_\phi}{mcz_o^2}\sqrt{\frac{3}{c^2+1}} = \sqrt{3}\dot{\phi}_o\frac{c}{\sqrt{c^2+1}} = \sqrt{3}\dot{\phi}_o\sin\alpha$$

where for the second equality I used (13.32) to replace p_ϕ by $mc^2z_o^2\dot{\phi}_o$ and for the third I replaced $c/\sqrt{c^2+1}$ by $\sin\alpha$ where α is the half-angle of the cone.

(d) For ω to be equal to $\dot{\phi}_o$, it must be that $\sin\alpha = 1/\sqrt{3}$ or $\alpha = 35.3°$. If this is the case, the height z will return to its initial value in the time for one complete orbit around the cone. Therefore the orbit is closed and is (approximately at least) a circle, tilted at a small angle $\epsilon_{\max}/cz_o$.

SECTION 13.4 Ignorable Coordinates

13.19 ⋆ In Example 13.3 (page 531) we saw that if we write the Hamiltonian for a two-dimensional central force problem in terms of polar coordinates r and ϕ, then the coordinate ϕ is ignorable. Write down the Hamiltonian for the same problem, but using rectangular coordinates x, y. Show that, with this choice, neither coordinate is ignorable. [The moral of this is that the choice of generalized coordinates calls for some care. In particular, you must look for any symmetries in a system and try to choose generalized coordinates to take advantage of them.]

The KE is $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ and the PE is $U = U(r) = U\left(\sqrt{x^2+y^2}\right)$. Thus the Hamiltonian is

$$\mathcal{H} = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + U\left(\sqrt{x^2+y^2}\right).$$

Clearly neither $\partial\mathcal{H}/\partial x$ nor $\partial\mathcal{H}/\partial y$ is zero (unless U is a constant); that is, neither x nor y is ignorable.

13.21 ⋆⋆ Two masses m_1 and m_2 are joined by a massless spring (force constant k and natural length l_o) and are confined to move in a frictionless horizontal plane, with CM and relative positions $\mathbf{R}$ and $\mathbf{r}$ as defined in Section 8.2. **(a)** Write down the Hamiltonian $\mathcal{H}$ using as generalized coordinates X, Y, r, ϕ, where (X, Y) are the rectangular components of $\mathbf{R}$, and (r, ϕ) are the polar coordinates of $\mathbf{r}$. Which coordinates are ignorable and which are not? Explain. **(b)** Write down the 8 Hamilton equations of motion. **(c)** Solve the r equations for the special case that $p_\phi = 0$ and describe the motion. **(d)** Describe the motion for the case that $p_\phi \neq 0$ and explain physically why the r equation is harder to solve in this case.

(a) The KE is $T = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2)$, and the PE is $U = \frac{1}{2}k(r - l_o)^2$. The generalized momenta are $\mathbf{P} = M\dot{\mathbf{R}}$, $p_r = \mu\dot{r}$, and $p_\phi = \mu r^2\dot{\phi}$, so the Hamiltonian is

$$\mathcal{H} = \frac{1}{2M}(P_x^2 + P_y^2) + \frac{1}{2\mu}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) + \tfrac{1}{2}k(r - l_o)^2.$$

The CM coordinates X and Y are ignorable, because there are no external forces, so that total momentum is conserved. The coordinate ϕ is ignorable because the force between the two masses is central, so their angular momentum is conserved. The coordinate r is not ignorable, because there is a radial force.

(b) The two Hamilton equations for X are

$$\dot{X} = \frac{\partial \mathcal{H}}{\partial P_x} = \frac{P_x}{M} \qquad \text{and} \qquad \dot{P}_x = -\frac{\partial \mathcal{H}}{\partial X} = 0$$

with corresponding equations for Y. These four equations say that the CM position $\mathbf{R}$ moves like a free particle, with constant velocity. The two equations for r are

$$\dot{r} = \frac{\partial \mathcal{H}}{\partial p_r} = \frac{p_r}{\mu} \qquad \text{and} \qquad \dot{p}_r = -\frac{\partial \mathcal{H}}{\partial r} = \frac{p_\phi^2}{\mu r^3} - k(r - l_o),$$

and the two ϕ equations are

$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{\mu r^2} \qquad \text{and} \qquad \dot{p}_\phi = -\frac{\partial \mathcal{H}}{\partial \phi} = 0$$

which last says that p_ϕ is conserved as expected.

(c) The two r equations combine to give the familiar radial equation

$$\mu\ddot{r} = \dot{p}_r = \frac{p_\phi^2}{\mu r^3} - k(r - l_o). \tag{S13.7}$$

In the special case that $p_\phi = 0$ this shows that r executes SHM about the equilibrium length $r = l_o$. The CM moves with constant velocity while the two masses oscillate toward and away from the CM.

(d) If $p_\phi \neq 0$, the radial equation (S13.7) is no longer linear and cannot be solved in terms of elementary functions. In this case, the two masses still oscillate in and out, but not in SHM, while they orbit around the CM with constant angular momentum.

13.23 ⋆⋆⋆ Consider the modified Atwood machine shown in Figure 13.11. The two weights on the left have equal masses m and are connected by a massless spring of force constant k. The weight on the right has mass $M = 2m$, and the pulley is massless and frictionless. The coordinate x is the extension of the spring from its equilibrium length; that is, the length of the spring is $l_e + x$ where l_e is the equilibrium length (with all the weights in position and M held stationary). **(a)** Show that the total potential energy (spring plus gravitational) is just $U = \frac{1}{2}kx^2$ (plus a constant that we can take to be zero). **(b)** Find the two momenta conjugate to x and y. Solve for $\dot{x}$ and $\dot{y}$, and write down the Hamiltonian. Show that the coordinate y is ignorable. **(c)** Write down the four Hamilton equations and solve them for the following initial conditions: You hold the mass M fixed with the whole system in equilibrium and $y = y_o$. Still holding M fixed, you pull the lower mass m down a distance x_o, and at $t = 0$ you let go of both masses. [*Hint:* Write down the initial values of x, y and their momenta. You can solve the x equations by combining them into a second-order equation for x. Once you know $x(t)$, you can quickly write down the other three variables.] Describe the motion. In particular, find the frequency with which x oscillates.

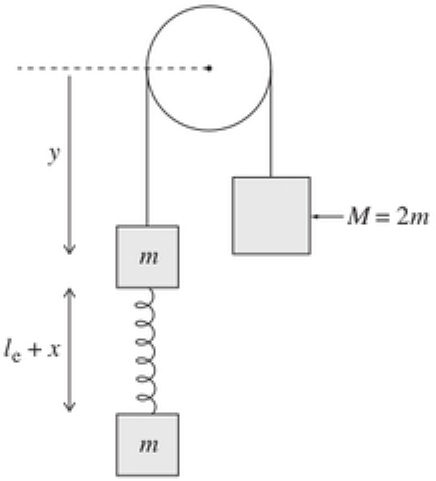

Figure 13.11 Problem 13.23

(a) The gravitational PE is $U_{\rm gr} = Mgy - mgy - mg(x+y) + \text{const} = -mgx$ if we drop the uninteresting constant. The spring PE is harder. If we let l_o denote the spring's natural, unloaded length, then $k(l_e - l_o) = mg$ and if x' denotes the spring's true extension (from its unloaded length), then $l_o + x' = l_e + x$ so

$$x' = x + (l_e - l_o) = x + \frac{mg}{k}$$

Thus the spring PE is

$$U_{\rm spr} = \tfrac{1}{2}kx'^2 = \tfrac{1}{2}k\left(x + \frac{mg}{k}\right)^2 = \tfrac{1}{2}kx^2 + mgx + \text{const.}$$

If we add this to $U_{\rm gr} = -mgx$, the terms in mgx cancel and (dropping another uninteresting constant) we get $U = U_{\rm gr} + U_{\rm spr} = \frac{1}{2}kx^2$ as claimed.

(b) The KE is $T = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m(\dot{x}+\dot{y})^2 = \frac{1}{2}m[3\dot{y}^2 + (\dot{x}+\dot{y})^2]$, from which we find the momenta,

$$p_x = \frac{\partial T}{\partial \dot{x}} = m(\dot{x}+\dot{y}) \qquad \text{and} \qquad p_y = \frac{\partial T}{\partial \dot{y}} = m(\dot{x}+4\dot{y})$$

whence

$$\dot{x} + \dot{y} = \frac{p_x}{m} \qquad \text{and} \qquad \dot{y} = \frac{1}{3m}(p_y - p_x).$$

From these we can calculate the Hamiltonian,

$$\mathcal{H} = T + U = \frac{1}{2m}\left[\frac{(p_x - p_y)^2}{3} + p_x^2\right] + \frac{1}{2}kx^2.$$

Because this doesn't depend on y, the coordinate y is ignorable. This is traceable to the fact that the total mass on each side is the same.

(c) The Hamilton equations for x are

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p_x} = \frac{1}{3m}(4p_x - p_y) \qquad \text{and} \qquad \dot{p}_x = -\frac{\partial \mathcal{H}}{\partial x} = -kx \tag{S13.8}$$

and those for y

$$\dot{y} = \frac{\partial \mathcal{H}}{\partial p_y} = \frac{1}{3m}(p_y - p_x) \qquad \text{and} \qquad \dot{p}_y = -\frac{\partial \mathcal{H}}{\partial y} = 0. \tag{S13.9}$$

The initial conditions are that $x(0) = x_o$, $y(0) = y_o$, and $\dot{x}(0) = \dot{y}(0) = 0$. These imply that $p_x(0) = p_y(0) = 0$, and, because p_y is constant, $p_y = 0$ for all time. Combining the two equations (S13.8) and setting $p_y = 0$, we find that $\ddot{x} = 4\dot{p}_x/3m = -4kx/3m$. Therefore $x = x_o \cos \omega t$, where $\omega = \sqrt{4k/3m}$. Next, from the first of Eqs. (S13.8) (with $p_y = 0$) we find that $p_x = \frac{3}{4}m\dot{x} = -\frac{3}{4}m\omega x_o \sin \omega t$ and finally, from the first of Eqs. (S13.9), $\dot{y} = -p_x/3m = \frac{1}{4}\omega x_o \sin \omega t$, so $y = -\frac{1}{4}x_o \cos \omega t$ + const = $y_o + \frac{1}{4}x_o(1 - \cos \omega t)$.

SECTION 13.5 Lagrange's Equations vs. Hamilton's Equations

13.25 ★★★ Here is another example of a canonical transformation, which is still too simple to be of any real use, but does nevertheless illustrate the power of these changes of coordinates. **(a)** Consider a system with one degree of freedom and Hamiltonian $\mathcal{H} = \mathcal{H}(q, p)$ and a new pair of coordinates Q and P defined so that

$$q = \sqrt{2P} \sin Q \qquad \text{and} \qquad p = \sqrt{2P} \cos Q. \tag{13.62}$$

Prove that if $\partial\mathcal{H}/\partial q = -\dot{p}$ and $\partial\mathcal{H}/\partial p = \dot{q}$, it automatically follows that $\partial\mathcal{H}/\partial Q = -\dot{P}$ and $\partial\mathcal{H}/\partial P = \dot{Q}$. In other words, the Hamiltonian formalism applies just as well to the new coordinates as to the old. **(b)** Show that the Hamiltonian of a one-dimensional harmonic oscillator with mass $m = 1$ and force constant $k = 1$ is $\mathcal{H} = \frac{1}{2}(q^2 + p^2)$. **(c)** Show that if you rewrite this Hamiltonian in terms of the coordinates Q and P defined in (13.62), then Q is ignorable. [The change of coordinates (13.62) was cunningly chosen to produce this elegant result.] What is P? **(d)** Solve the Hamiltonian equation for $Q(t)$ and verify that, when rewritten for q, your solution gives the expected behavior.

(a) The new variables Q and P are defined so that

$$q = \sqrt{2P} \sin Q \qquad \text{and} \qquad p = \sqrt{2P} \cos Q \tag{S13.10}$$

and we are promised that the old variables q and p satisfy Hamilton's equations. So now consider this:

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial Q} &= \frac{\partial \mathcal{H}}{\partial q}\frac{\partial q}{\partial Q} + \frac{\partial \mathcal{H}}{\partial p}\frac{\partial p}{\partial Q} = -\dot{p}(\sqrt{2P}\cos Q) - \dot{q}(\sqrt{2P}\sin Q) \\ &= -(\dot{p}p + \dot{q}q) = -\frac{1}{2}\frac{d}{dt}(p^2 + q^2) = -\frac{d}{dt}(P\cos^2 Q + P\sin^2 Q) = -\dot{P}.\end{aligned}$$

This proves the Hamilton equation for $\dot{P}$. Next

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial P} &= \frac{\partial \mathcal{H}}{\partial q}\frac{\partial q}{\partial P} + \frac{\partial \mathcal{H}}{\partial p}\frac{\partial p}{\partial P} = -\dot{p}\left(\frac{1}{\sqrt{2P}}\sin Q\right) + \dot{q}\left(\frac{1}{\sqrt{2P}}\cos Q\right) \\ &= -\dot{p}\frac{q}{2P} + \dot{q}\frac{p}{2P} = \frac{p^2}{2P}\frac{d}{dt}\left(\frac{q}{p}\right) = \frac{p^2}{2P}\frac{d}{dt}\tan Q = \frac{p^2}{2P}\dot{Q}\sec^2 Q = \dot{Q}.\end{aligned}$$

Here in moving to the second line I used Eqs. (S13.10), and for the final equality I used the second of Eqs. (S13.10). This proves the Hamilton equation for $\dot{Q}$.

(b) $\mathcal{H} = (p^2/2m) + \frac{1}{2}kq^2 = \frac{1}{2}(p^2 + q^2)$ because $m = k = 1$.

(c) $\mathcal{H} = \frac{1}{2}(p^2 + q^2) = \frac{1}{2}(2P\cos^2 Q + 2P\sin^2 Q) = P$. We see that Q is ignorable and that the new Hamiltonian is just P (so that conservation of P is just conservation of energy).

(d) The Hamilton equation for Q reads $\dot{Q} = \partial\mathcal{H}/\partial P = 1$, which implies that $Q = (t - \delta)$ where δ is an arbitrary constant. Substituting this into the first of Eqs. (S13.10) and putting $P = E$, we find $q = \sqrt{2E}\sin(t - \delta)$, which correctly describes an SHO with energy E and frequency $\omega = \sqrt{k/m} = 1$.

SECTION 13.6 Phase-Space Orbits

13.27 ⋆⋆ Figure 13.6 shows some phase-space orbits for a mass in free fall. The points A_o, B_o, C_o, D_o represent four different possible initial states at time 0, and A, B, C, D are the corresponding states at a later time. Write down the position $x(t)$ and momentum $p(t)$ as functions of t and use these to prove that $ABCD$ is a parallelogram with area equal to the rectangle $A_oB_oC_oD_o$. [This is an example of Liouville's theorem.]

The position and momentum of an object in free fall are $x = x_o + (p_o/m)t + \frac{1}{2}gt^2$ and $p = p_o + mgt$. Thus the positions in phase space of the eight points in Fig.13.6 are as follows:

	time 0			time t	
	x_o	p_o		x	p
A_o	0	0	A	$\frac{1}{2}gt^2$	mgt
B_o	X	0	B	$X + \frac{1}{2}gt^2$	mgt
C_o	X	P	C	$X + (P/m)t + \frac{1}{2}gt^2$	$P + mgt$
D_o	0	P	D	$(P/m)t + \frac{1}{2}gt^2$	$P + mgt$

Inspecting the data in this table, you can see the heights of the two points D_o and C_o above the x axis are both equal to P, so the lines D_oC_o and A_oB_o are parallel. Since both of these lines have the same length (X), $A_oB_oC_oD_o$ is a parallelogram. (In fact it is also a rectangle,but this doesn't matter here.) In the same way, the lines AB and DC are parallel and of equal length, so $ABCD$ is also a parallogram. Again from the data in the table you can check that both parallograms have bases equal to X and heights equal to P. Thus both have area $X \cdot P$. In particular, their areas are the same.

SECTION 13.7 Liouville's Theorem*

13.29 ⋆ Figure 13.10 shows an initially spherical volume getting stretched into an ellipsoid by the shearing flow (13.57). Make a similar sketch for a volume that is initially spherical and centered on the origin.

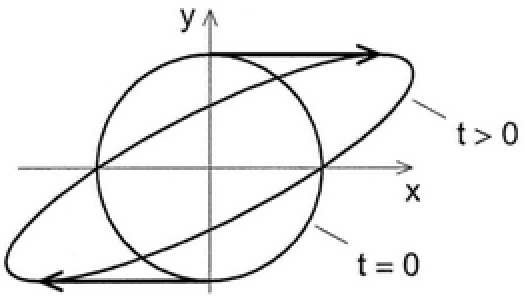

Figure S13.29

13.31 ⋆ Evaluate the three-dimensional divergence $\nabla \cdot \mathbf{v}$ for each of the following vectors: **(a)** $\mathbf{v} = k\mathbf{r}$, **(b)** $\mathbf{v} = k(z, x, y)$, **(c)** $\mathbf{v} = k(z, y, x)$, **(d)** $\mathbf{v} = k(x, y, -2z)$, where $\mathbf{r} = (x, y, z)$ is the usual position vector and k is a constant.

(a) If $\mathbf{v} = k\mathbf{r} = (kx, ky, kz)$, then $\nabla \cdot \mathbf{v} = k\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) = 3k$

(b) If $\mathbf{v} = k(z, x, y)$, then $\nabla \cdot \mathbf{v} = k\left(\frac{\partial z}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial y}{\partial z}\right) = 0$

(c) If $\mathbf{v} = k(z, y, x)$, then $\nabla \cdot \mathbf{v} = k\left(\frac{\partial z}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial x}{\partial z}\right) = k$

(d) If $\mathbf{v} = k(x, y, -2z)$, then $\nabla \cdot \mathbf{v} = k\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} - 2\frac{\partial z}{\partial z}\right) = k(1 + 1 - 2) = 0$

13.33 ⋆⋆ The divergence theorem is a remarkable result, relating the *surface* integral that gives the flow of **v** out of a closed surface S to the *volume* integral of $\nabla \cdot \mathbf{v}$. Occasionally it is easy to evaluate both of these integrals and one can check the validity of the theorem. More often, one of the integrals is much easier to evaluate than the other, and the divergence theorem then gives one a slick way to evaluate a hard integral. The following exercises illustrate both of these situations. **(a)** Let $\mathbf{v} = k\mathbf{r}$, where k is a constant and let S be a sphere of radius R centered on the origin. Evaluate the left side of the divergence theorem (13.56) (the surface integral). Next calculate $\nabla \cdot \mathbf{v}$ and use this to evaluate the right side of (13.56) (the volume integral). Show that the two agree. **(b)** Now use the same velocity **v**, but let S be a sphere *not* centered on the origin. Explain why the surface integral is now hard to evaluate directly, but don't actually do it. Instead, find its value by doing the volume integral. (This second route should be no harder than before.)

The divergence theorem asserts that

$$\int_S \mathbf{v} \cdot \mathbf{n}\, dA = \int_V \nabla \cdot \mathbf{v}\, dV. \tag{S13.11}$$

(a) If S is a sphere of radius R centered on the origin, then the unit normal **n** points radially out from the origin, so $\mathbf{n} = \hat{\mathbf{r}}$, and the surface integral on the left side of Eq. (S13.11) is easily evaluated (with $\mathbf{v} = k\mathbf{r}$):

$$\text{LHS} = \int_S \mathbf{v} \cdot \mathbf{n}\, dA = \int_S k\mathbf{r} \cdot \hat{\mathbf{r}}\, dA = kR \int_S dA = 4\pi k R^3. \tag{S13.12}$$

Because $\nabla \cdot \mathbf{v} = 3k$, the volume integral on the right side of Eq. (S13.11) is easily seen to be

$$\text{RHS} = \int_V \nabla \cdot \mathbf{v}\, dV = 3k \int_V dV = 3k\left(\frac{4}{3}\pi R^3\right) = 4\pi k R^3 \tag{S13.13}$$

in agreement with (S13.12). That is, this example confirms the divergence theorem.

(b) If S is not centered on the origin, then the normal **n** is not the same as $\hat{\mathbf{r}}$ and the integrand $\mathbf{v} \cdot \mathbf{n}$ in the LHS of Eq. (S13.11) is a somewhat complicated function of position on the sphere. This makes the surface integral harder to evaluate directly. On the other hand, the volume integral in the RHS can be evaluated exactly as before in (S13.13). Therefore, the surface integral has the same value as before, $4\pi k R^3$.

13.35 ⋆⋆ A beam of particles is moving along an accelerator pipe in the z direction. The particles are uniformly distributed in a cylindrical volume of length L_o (in the z direction) and radius R_o. The particles have momenta uniformly distributed with p_z in an interval $p_o \pm \Delta p_z$ and the transverse momentum $p_\perp$ inside a circle of radius $\Delta p_\perp$. To increase the particles' spatial density, the beam is focused by electric and magnetic fields, so that the radius shrinks to a smaller value R. What does Liouville's theorem tell you about the spread in the transverse momentum $p_\perp$ and the subsequent behavior of the radius R? (Assume that the focusing does not affect either L_o or Δp_z.)

The initial volume occupied by the beam is $V_o = (\pi R_o^2 L_o)[\pi(\Delta p_\perp)^2 2\Delta p_z]$. (This is the volume in phase space.) By Liouville's theorem, this volume can't change. Thus when R shrinks (with L_o and Δp_z fixed), $\Delta p_\perp$ has to grow. In the long run, this means that R will increase again.

13.37 ⋆⋆⋆ The general proof of the divergence theorem

$$\int_S \mathbf{n} \cdot \mathbf{v}\, dA = \int_V \nabla \cdot \mathbf{v}\, dV \tag{13.63}$$

is fairly complicated and not especially illuminating. However, there are a few special cases where it is reasonably simple and quite instructive. Here is one: Consider a rectangular region bounded by the six planes $x = X$ and $X + A$, $y = Y$ and $Y + B$, and $z = Z$ and $Z + C$, with total volume $V = ABC$. The surface S of this region is made up of six rectangles that we can call S_1 (in the plane $x = X$), S_2 (in the plane $x = X + A$), and so on. The surface integral on the left of (13.63) is then the sum of six integrals, one over each of the rectangles S_1, S_2, and so forth. **(a)** Consider the first two of these integrals and show that

$$\int_{S_1} \mathbf{n} \cdot \mathbf{v}\, dA + \int_{S_2} \mathbf{n} \cdot \mathbf{v}\, dA = \int_Y^{Y+B} dy \int_Z^{Z+C} dz\ [v_x(X + A, y, z) - v_x(X, y, z)\ ].$$

(b) Show that the integrand on the right can be rewritten as an integral of $\partial v_x/\partial x$ over x running from $x = X$ to $x = X + A$. **(c)** Substitute the result of part (b) into part (a), and write down the corresponding results for the two remaining pairs of faces. Add these results to prove the divergence theorem (13.63).

(a) As discussed, the surface integral on the left side of Eq. (13.63) has the form $I_1 + \cdots + I_6$, where $I_1 = \int_{S_1} \mathbf{n} \cdot \mathbf{v}\, dA$ and so on. Now, on the surface S_1 the outward normal is $\mathbf{n} = -\hat{\mathbf{x}}$, while on S_2, $\mathbf{n} = \hat{\mathbf{x}}$. Therefore,

$$\begin{aligned} I_1 + I_2 &= -\int_Y^{Y+B} dy \int_Z^{Z+C} dz\, v_x(X, y, z) + \int_Y^{Y+B} dy \int_Z^{Z+C} dz\, v_x(X + A, y, z) \\ &= \int_Y^{Y+B} dy \int_Z^{Z+C} dz\, [v_x(X + A, y, z) - v_x(X, y, z)]. \end{aligned} \tag{S13.14}$$

(b) By the fundamental theorem of calculus, $f(X + A) - f(X) = \int_X^{X+A} \frac{df}{dx} dx$, so the integrand in (S13.14) is $\int_X^{X+A} \frac{\partial v_x}{\partial x} dx$.

(c) Combining the results of parts (a) and (b), we see that

$$I_1 + I_2 = \int_X^{X+A} dx \int_Y^{Y+B} dy \int_Z^{Z+C} dz\, \frac{\partial v_x}{\partial x} = \int_V \frac{\partial v_x}{\partial x} dV.$$

Adding this to the corresponding results for the other two pairs of integrals, we conclude that the surface integral of (13.63) is

$$\int_S \mathbf{v} \cdot \mathbf{n}\, dA = I_1 + \cdots + I_6 = \int_V \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) dV = \int_V \nabla \cdot \mathbf{v}\, dV$$

which is the divergence theorem.

CHAPTER 14

Collision Theory

PROBLEMS AND SOLUTIONS

SECTION 14.2 The Collision Cross Section

14.1 ⋆ A blueberry pancake has diameter 15 cm and contains 6 large blueberries, each of diameter 1 cm. Find the cross section σ of a blueberry and the "target" density $n_{\rm tar}$ (number/area) of berries in the pancake, as seen from above. What is the probability that a skewer, jabbed at random into the pancake, will hit a berry (in terms of σ and $n_{\rm tar}$, and then numerically)?

The area of a blueberry is $\sigma = \pi r^2 = \pi d^2/4 = \pi/4$ cm^2 = 0.79 cm^2. The density of targets (number/area) is $n_{\rm tar} = N_{\rm tar}/A = N_{\rm tar}/(\pi D^2/4) = 0.034$ cm^{-2}. The probability of a hit in one try is $n_{\rm tar}\sigma = 0.027$.

14.3 ⋆ A beam of particles is directed through a tank of liquid hydrogen. If the tank's length is 50 cm and the liquid density is 0.07 gram/cm^3, what is the target density (number/area) of hydrogen atoms seen by the incident particles?

The density is $\varrho = 0.07$ g/cm^3, the length of the tank, $L = 50$ cm, and the mass of an H atom is $m_{\rm H} = 1.66 \times 10^{-27}$ kg. The number density (number/volume) is $\varrho/m_{\rm H}$, so the target density (number/area) is

$$n_{\rm tar} = (\text{number density}) \times L = \frac{\varrho L}{m_{\rm H}} = \frac{(0.07 \times 10^3\ \text{kg/m}^3) \times (0.5\ \text{m})}{1.66 \times 10^{-27}\ \text{kg}} \approx 2.1 \times 10^{28}\ \text{atoms/m}^2$$

14.5 ⋆⋆ The cross section for scattering a certain nuclear particle by a nitrogen nucleus is 0.5 barns. If 10^{11} of these particles are fired through a cloud chamber of length 10 cm, containing nitrogen at STP, how many particles are scattered? (Use the ideal gas law and remember that each nitrogen molecule has two atoms. The scattering by any atomic electrons is completely negligible.)

The target density is $n_{\rm tar}$ = (number density) × thickness, where in an ideal gas of nitrogen the number density is $2 \times (6.02 \times 10^{23})/(22.4\ \text{liters}) = 5.38 \times 10^{25}$ particles/m^3. (Remember that a mole of gas occupies 22.4 liters and that each N_2 molecule has two atoms.) Thus $n_{\rm tar} = 5.38 \times 10^{24}$ particles/m^2. Therefore

$$N_{\rm sc} = N_{\rm inc} n_{\rm tar} \sigma = 10^{11} \times (5.38 \times 10^{24}\ \text{m}^{-2}) \times (0.5 \times 10^{-28}\ \text{m}^2) = 2.7 \times 10^7\ \text{particles.}$$

SECTION 14.4 The Differential Scattering Cross Section

14.7 ⋆ Calculate the solid angles subtended by the moon and by the sun, both as seen from the earth. Comment on your answers. (The radii of the moon and sun are $R_{\rm m} = 1.74 \times 10^6$ m and $R_{\rm s} = 6.96 \times 10^8$ m. Their distances from earth are $d_{\rm m} = 3.84 \times 10^8$ m and $d_{\rm s} = 1.50 \times 10^{11}$ m.)

A sphere of radius R at a large distance d subtends a solid angle $\Delta\Omega \approx A/d^2 = \pi R^2/d^2$. For the moon this gives $\Delta\Omega_{\rm moon} \approx 6.45 \times 10^{-5}$ sr, and for the sun, $\Delta\Omega_{\rm sun} \approx 6.76 \times 10^{-5}$ sr. Because $\Delta\Omega_{\rm moon} \approx \Delta\Omega_{\rm sun}$, the moon and sun *appear* to be about the same size.

14.9 ⋆ By integrating the element of solid angle (14.15), $d\Omega = \sin\theta\, d\theta\, d\phi$, over all directions, verify that the solid angle corresponding to all directions is 4π steradians.

$$\Omega = \int d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi} (\sin\theta) d\theta = 2\pi[-\cos\theta]_0^{\pi} = 4\pi.$$

14.11 ⋆⋆ The differential cross section for scattering 6.5-MeV alpha particles at 120° off a silver nucleus is about 0.5 barns/sr. If a total of 10^{10} alphas impinge on a silver foil of thickness 1 μm and if we detect the scattered particles using a counter of area 0.1 mm^2 at 120° and 1 cm from the target, about how many scattered alphas should we expect to count? (Silver has a specific gravity of 10.5, and atomic mass of 108.)

The target density is

$$n_{\rm tar} = \frac{\varrho t}{m_{\rm Ag}} = \frac{(10.5 \times 10^3\ {\rm kg/m^3}) \times (10^{-6}\ {\rm m})}{108 \times (1.66 \times 10^{-27}\ {\rm kg})} = 5.86 \times 10^{22}\ {\rm m^{-2}}$$

and the solid angle subtended by the counter is $\Delta\Omega = (0.1\ {\rm mm^2})/(10\ {\rm mm})^2 = 10^{-3}$ sr. Therefore, the number of alphas scattered into $\Delta\Omega$ should be

$$N_{\rm sc} = N_{\rm inc} n_{\rm tar} \frac{d\sigma}{d\Omega} \Delta\Omega = (10^{10}) \times (5.86 \times 10^{22}\ {\rm m^{-2}}) \times (0.5 \times 10^{-28}\ {\rm m^2/sr}) \times (10^{-3}\ {\rm sr}) \approx 29.$$

SECTION 14.5 Calculating the Differential Cross Section

14.13 ⋆⋆ In deriving the cross section for scattering by a hard sphere, we used the "law of reflection," that the angles of incidence and reflection of a particle bouncing off a hard sphere are equal, as in Figure 14.10. Use conservation of energy and angular momentum to prove this law. (The definition of "hard-sphere scattering" is that a projectile bounces with its kinetic energy unchanged. That the force is spherically symmetric implies, as usual, that angular momentum about the sphere's center is conserved.)

In Fig.14.10, let us temporarilly rename as α' the second of the angles labelled α (the angle of reflection), and let the incoming and outgoing speeds be v and v'. That the collision is elastic implies that $v = v'$, and conservation of angular momentum implies that $mvR\sin\alpha = mv'R\sin\alpha'$. Together, these imply that $\sin\alpha = \sin\alpha'$ and hence that $\alpha = \alpha'$.

14.15 ⋆⋆⋆ [Computer] Consider a point projectile moving in a fixed, spherical force whose potential energy is

$$U(r) = \begin{cases} -U_o & (0 \le r \le R) \\ 0 & (R < r) \end{cases} \tag{14.59}$$

where U_o is a positive constant. This so-called spherical well represents a projectile which moves freely in either of the regions $r < R$ and $R < r$, but, when it crosses the boundary $r = R$, receives a radially inward impulse that changes its kinetic energy by $\pm U_o$ ($+U_o$ going inward, $-U_o$ going outward). **(a)** Sketch the orbit of a projectile that approaches the well with momentum p_o and impact parameter $b < R$. **(b)** Use conservation of energy to find the momentum p of the projectile inside the well ($r < R$). Let ζ denote the momentum ratio $\zeta = p_o/p$ and let d denote the projectile's distance of closest approach to the origin. Use conservation of angular momentum to show that $d = \zeta b$. **(c)** Use your sketch to prove that the scattering angle θ is

$$\theta = 2\left(\arcsin\frac{b}{R} - \arcsin\frac{\zeta b}{R}\right). \tag{14.60}$$

This gives θ as a function of b, which is what you need to get the cross section. The relation depends on the momentum ratio ζ, which in turn depends on the incoming momentum p_o and the well depth U_o. Plot θ as a function of b for the case that $\zeta = 0.5$. **(d)** By differentiating θ with respect to b, find an expression for the differential cross section as a function of b, and make a plot of $d\sigma/d\Omega$ against θ for the case that $\zeta = 0.5$. Comment. [*Hint:* To plot as a function of θ you don't need to solve for b in terms of θ; instead, you can make a parametric plot of the point $(\theta, d\sigma/d\Omega)$ as a function of the parameter b running from 0 to R.] **(e)** By integrating $d\sigma/d\Omega$ over all directions, find the total cross section.

(a)

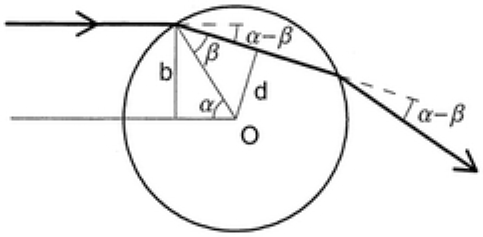

Figure S14.15a

Note that the angles shown are $\alpha = \arcsin(b/R)$ and $\beta = \arcsin(d/R)$, and the angle of deflection on entering the sphere is $\alpha - \beta$. By symmetry the deflection on leaving the sphere is the same, so the total angle of deflection is $\theta = 2(\alpha - \beta)$.

(b) The projectile's initial energy is $\frac{1}{2}mv_o^2 = p_o^2/2m$, while its energy inside the sphere is $p^2/2m - U_o$. Equating these two, we find that $p = \sqrt{p_o^2 + 2mU_o}$. The initial angular momentum is bp_o and that when inside the sphere is dp. Equating these we find that $d = bp_o/p = \zeta b$.

(c) From part (a), it is clear that $\theta = 2(\alpha - \beta) = 2[\arcsin(b/R) - \arcsin(\zeta b/R)]$, as in Equation (14.60). This is zero when $b = 0$ (the projectile goes straight through, undeflected). As b increases from 0 to R, the scattering angle θ increases steadily to a maximum, $\theta_{\max} = \pi - 2\arcsin\zeta$ when $b = R$, then drops abruptly to 0 and remains zero for all $b > R$ because the projectile misses the target entirely.

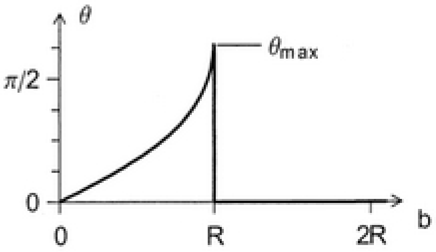

Figure S14.15c

(d) Differentiating (14.60), we find

$$\frac{d\sigma}{d\Omega} = \frac{b}{(\sin\theta)d\theta/db} \quad \text{where} \quad \frac{d\theta}{db} = 2\left(\frac{1}{\sqrt{R^2-b^2}} - \frac{\zeta}{\sqrt{R^2-\zeta^2 b^2}}\right).$$

This is good for $0 \le b \le R$ (equivalently $0 \le \theta \le \theta_{\max}$); for $\theta > \theta_{\max}$, $d\sigma/d\Omega = 0$.

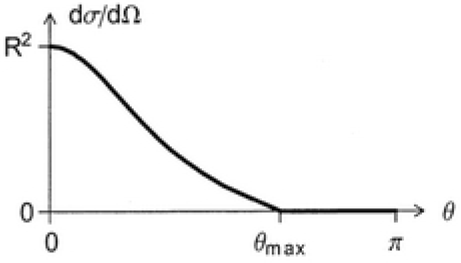

Figure S14.15d

(e) $\sigma_{\text{tot}} = 2\pi\int_0^\pi \frac{d\sigma}{d\Omega}\sin\theta\, d\theta = 2\pi\int_0^\pi \frac{b}{(\sin\theta)d\theta/db}\sin\theta\, d\theta = 2\pi\int_0^R b\, db = \pi R^2.$

SECTION 14.6 Rutherford Scattering

14.17 ⋆ Another specific prediction of Rutherford's model of the atom was that the cross section should be proportional to the nuclear charge squared, that is, to Z^2, where Z is the atomic number, the number of protons in the nucleus. To test this, Geiger and Marsden counted the number of scatterings off various different targets (holding all other variables fixed), with the following results:

Target	Gold	Platinum	Tin	Silver	Copper	Aluminum
N_{sc}	1319	1217	467	420	152	26
Z	79	78	50	47	29	13

Add a row to this table to show the ratio N_{sc}/Z^2 and see how well Rutherford's prediction was confirmed. (At the time the atomic number was not known with certainty, nor was it well understood. Rutherford had guessed, correctly, that the nuclear charge was roughly equal to half the atomic mass, and this is what they used in place of Z.) The relatively poor agreement for the case of aluminum is probably due to our neglect of the target recoil, which is more important for the lighter targets.)

Target	Gold	Platinum	Tin	Silver	Copper	Aluminum
N_{sc}:	1319	1217	467	420	152	26
Z:	79	78	50	47	29	13
N_{sc}/Z^2 :	0.21	0.20	0.19	0.19	0.18	0.15

14.19 ⋆⋆ An important simplification in our derivation of the Rutherford cross section was that the projectile's orbit is symmetric about the direction **u** of closest approach. (See Figure 14.11.) Prove that this is true of almost any conservative central force, as follows: **(a)** Assume that the effective potential (actual plus centrifugal) behaves as in Figure 8.4; that is, it approaches zero as $r \to \infty$ and approaches $+\infty$ as $r \to 0$.[1] Use this to prove that any projectile that comes in from infinity must reach a minimum value r_{min} and then move out to infinity again. **(b)** This implies that a projectile must visit any value of r between r_{min} and infinity exactly twice, once on the way in and again on the way out. Prove that the values of $\dot{r}$ at these two points are equal and opposite, and that the values of $\dot{\psi}$ are equal (where ψ is the polar angle defined in Figure 14.11). **(c)** Use these results to prove that the orbit is symmetric under reflection about the direction **u**.

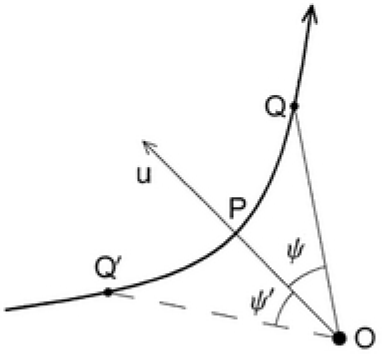

Figure S14.19

The figure shows a typical orbit of a projectile. The point O is the force center, P is the point of closest approach, and let us choose the time $t = 0$ to be the time at which the projectile passes through P. The point Q is the projectile's position at an arbitrary time t and Q' the position at time $-t$. To prove that the orbit is symmetric about the line OP, I shall prove that the points Q and Q' are equidistant from O (that is, $OQ = OQ'$) and make equal angles with the direction OP (that is, $\psi = \psi'$). Since Q is any point on the orbit, this does it.

(a) Under the assumptions on the effective PE, the projectile must have $E \geq 0$, corresponding to the upper dashed line in Fig.8.5. Since $U_{\text{eff}}(r) \to \infty$ as $r \to 0$, there has to be an r_{min} at which $E = U_{\text{eff}}(r_{\text{min}})$. The projectile moves steadily inward from $r = \infty$ until it reaches r_{min} and then moves steadily outward toward $r = \infty$ again. The distance r_{min} defines the point of closest approach labeled P in the figure here.

(b) Since $E = \frac{1}{2}m\dot{r}^2 + U_{\text{eff}}(r)$, it follows that $\dot{r}^2$ is a single-valued function of r on any one orbit. Therefore, the magnitude of $\dot{r}$ at any distance r on the inward trip is the same as that at the same distance r on the outward trip: $\dot{r}(r)_{\text{in}} = -\dot{r}(r)_{\text{out}}$. It immediately follows that the time to move in from r to r_{min} is the same as that to move out again from r_{min} to r. Turning this around, if the times from Q' to P and from P to Q are equal, then the distances OQ' and OQ are equal.

Using conservation of angular momentum ℓ, we can prove a similar result for the angle ψ. Since $\ell = mr^2\dot{\psi}$, it follows that $\dot{\psi}$ is a single-valued function of r. Therefore $\dot{\psi}$ is the same at any point Q and at the corresponding Q', which in turn means that the change in ψ from Q' to P is the same as that from P to Q. That is $\psi = \psi'$.

(c) The symmetry of the orbit now follows as in the first paragraph above.

14.20 ⋆⋆ The derivation of the Rutherford cross section was made simpler by the fortuitous cancellation of the factors of r in the integral (14.30). Here is a method of finding the cross section which works, in principle, for any central force field: The general appearance of the scattering orbit is as shown in Figure 14.11. It is symmetric about the direction **u** of closest approach. (See Problem 14.19.) If ψ is the projectile's polar angle, measured from the direction **u**, then $\psi \to \pm\psi_o$ as $t \to \pm\infty$ and the scattering angle is $\theta = \pi - 2\psi_o$. (See Figure 14.11 again.) The angle ψ_o is equal to $\int \dot\psi\, dt$ taken from the time of closest approach to ∞. Using the now-familiar trick you can rewrite this as $\int (\dot\psi/\dot r)\, dr$. Next rewrite $\dot\psi$ in terms of the angular momentum ℓ and r, and rewrite $\dot r$ in terms of the energy E and the effective potential U_{eff} defined in Equation (8.35). Having done all this you should be able to prove that

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[1] Although this behavior is definitely the norm, there are a few force fields for which it is not true. If the actual potential energy is strongly attractive near $r = 0$ (for example, $U(r) = -1/r^3$), then it dominates the centrifugal potential near $r = 0$, and the effective potential does not approach $+\infty$ as $r \to 0$. Our argument also breaks down for the special case that $b = 0$, in which case the projectile may smash directly into the target.

$$\theta = \pi - 2\int_{r_{\min}}^{\infty} \frac{(b/r^2)\, dr}{\sqrt{1-(b/r)^2 - U(r)/E}}. \tag{14.61}$$

Provided this integral can be evaluated, it gives θ in terms of b, and hence the cross section (14.23). For examples of its use, see Problems 14.21, 14.22, and 14.23.

14.21 ⋆⋆ Use the general relation (14.61) from Problem 14.20 to rederive the relation (14.24) for scattering by a hard sphere.

For a hard sphere, $r_{\min} = R$ and, since $U(r) = 0$ for $r > R$, Eq. (14.61) reduces to

$$\theta = \pi - 2\int_R^{\infty} \frac{(b/r^2)dr}{\sqrt{1-(b/r)^2}} = \pi + 2\int_{b/R}^{0} \frac{du}{\sqrt{1-u^2}} = \pi - 2\arcsin(b/R),$$

where to get the second integral I substituted $u = b/r$. This is quickly solved to give $b = R\cos(\theta/2)$ as in (14.24).

14.23 ⋆⋆⋆ Consider the scattering of a particle with energy E by a fixed, repulsive $1/r^3$ force field, with potential energy $U = \gamma/r^2$. Use the relation (14.61) from Problem 14.20 to find θ in terms of b and hence show that the differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\gamma}{E}\,\frac{\pi^2(\pi-\theta)}{\theta^2(2\pi-\theta)^2\sin\theta}. \tag{14.62}$$

To refresh your memory as to how to find $r_{\min}$ you might look at Figure 8.5 (the case $E > 0$). You should be able to solve your equation for θ in terms of b to get b in terms of θ and thence the cross section.

Before we use Eq. (14.61) we need to find $r_{\min}$. This is determined by the condition that $U_{\text{eff}} = E$ where

$$U_{\text{eff}} = U + \frac{\ell^2}{2mr^2} = \frac{\gamma}{r^2} + \frac{p^2b^2}{2mr^2} = \left(\gamma + Eb^2\right)\frac{1}{r^2}.$$

Therefore, the minimum value of r (where $U_{\text{eff}} = E$) is

$$r_{\min} = \sqrt{b^2 + \gamma/E}.$$

Notice that this let's us rewrite U_{eff} as $U_{\text{eff}} = E\,(r_{\min}/r)^2$.

Returning to Eq. (14.61), we can write

$$\pi - \theta = 2\int_{r_{\min}}^{\infty} \frac{(b/r^2)\, dr}{\sqrt{1-U_{\text{eff}}(r)/E}} = 2b\int_{r_{\min}}^{\infty} \frac{dr/r^2}{\sqrt{1-(r_{\min}/r)^2}} = \frac{\pi b}{r_{\min}} = \frac{\pi b}{\sqrt{b^2+\gamma/E}},$$

which we can solve to give $b^2 = \frac{\gamma}{E} \cdot \frac{(\pi - \theta)^2}{\theta(2\pi - \theta)}$. Finally, as you can easily check,

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{1}{2\sin\theta}\left|\frac{d(b^2)}{d\theta}\right| = \frac{\gamma}{E}\frac{\pi^2(\pi - \theta)}{\theta^2(2\pi - \theta)^2\sin\theta}.$$

SECTION 14.8 Relation of the CM and Lab Scattering Angles

14.25 ⋆⋆ Using Figure 14.15, prove Equation (14.53), that

$$\tan\theta_{\text{lab}} = \frac{\sin\theta_{\text{cm}}}{\lambda + \cos\theta_{\text{cm}}}. \tag{14.64}$$

Consider the right triangle ADF, whose height is $DF = p'\sin\theta_{\text{cm}} = p\sin\theta_{\text{cm}}$ (recall that $p = p'$) and whose base is $AF = \lambda p + p\cos\theta_{\text{cm}}$. Therefore

$$\tan\theta_{\text{lab}} = \frac{DF}{AF} = \frac{p\sin\theta_{\text{cm}}}{\lambda p + p\cos\theta_{\text{cm}}} = \frac{\sin\theta_{\text{cm}}}{\lambda + \cos\theta_{\text{cm}}},$$

as claimed.

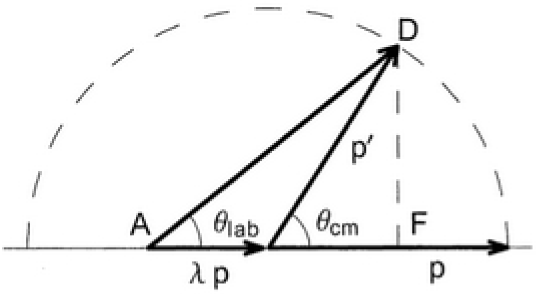

Figure S14.25

14.27 ⋆⋆ **(a)** By setting the mass ratio $\lambda = 1$ in Equation (14.64) of Problem 14.25, prove that in equal-mass scattering, $\theta_{\text{lab}} = \frac{1}{2}\theta_{\text{cm}}$. **(b)** Redraw Figure 14.15 for the case that $\lambda = 1$ ($m_1 = m_2$) and explain why the maximum value of θ_{lab} is $\pi/2$.

(a) With $\lambda = 1$, Eq. (14.64) reads

$$\tan\theta_{\text{lab}} = \frac{\sin\theta_{\text{cm}}}{1 + \cos\theta_{\text{cm}}} = \frac{2\sin(\theta_{\text{cm}}/2)\cos(\theta_{\text{cm}}/2)}{1 + \cos^2(\theta_{\text{cm}}/2) - \sin^2(\theta_{\text{cm}}/2)} = \frac{\sin(\theta_{\text{cm}}/2)}{\cos(\theta_{\text{cm}}/2)} = \tan(\theta_{\text{cm}}/2).$$

Therefore, $\theta_{\text{lab}} = \frac{1}{2}\theta_{\text{cm}}$.

(b) Compare the figure here with Figure 14.15. Here, with $\lambda = 1$, the point A has moved out and coincides with E. If $m_1 < m_2$ (Fig. 14.15), then as θ_{cm} increases toward π the point D moves around toward E and $\theta_{\text{lab}} \to \pi$. But here, with equal masses, as θ_{cm} increases toward π the line AD becomes tangent to the circle at A and $\theta_{\text{lab}} \to \pi/2$.

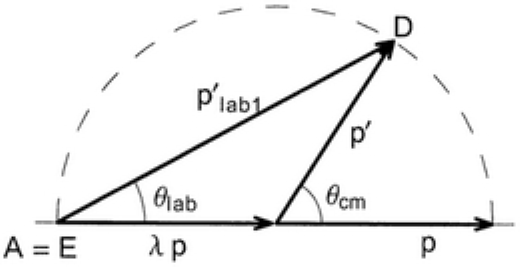

Figure S14.27

14.29 ⋆⋆ An elastic collision is defined as one in which the total kinetic energy of the two particles is the same before and after the collision. **(a)** Show that in the CM frame, the individual kinetic energies of the two particles are separately conserved in an elastic collision. **(b)** Explain clearly why the same result is obviously *not* true in the lab frame. (Think about the energy of the target particle.) **(c)** Let ΔE denote the energy gained by the target particle in the collision (and hence the energy lost by the projectile). Using Figure 14.15, show that the fractional energy lost by the projectile (in the lab frame) is

$$\frac{\Delta E}{E} = \frac{4\lambda}{(1+\lambda)^2}\sin^2(\theta_{\rm cm}/2)$$

where, as usual, λ is the mass ratio m_1/m_2. (Note that in Figure 14.15 the line DC represents the recoil momentum of the target.) **(d)** For a given mass ratio λ, what sort of collision gives the largest fractional energy loss? What value of λ maximizes this energy loss? (Your answer is important in situations where one wants a particle to lose energy as quickly as possible — as in a nuclear reactor, for example.)

(a) In the CM frame the total momentum is zero, so the two initial momenta are equal in magnitude, $p_1 = p_2$ and likewise the two final momenta, $p_1' = p_2'$. This let's us write the conservation of kinetic energy (elastic collision) as

$$E = p_1^2\left(\frac{1}{2m_1}+\frac{1}{2m_2}\right) = E' = p_1'^2\left(\frac{1}{2m_1}+\frac{1}{2m_2}\right)$$

which implies that $p_1 = p_1'$ and hence that the KE of particle 1 is separately conserved, and likewise for particle 2.

(b) In the lab frame, particle 2 is initially at rest, so that $T_2 = 0$. If any kind of collision occurs, particle 2 must recoil with $T_2 > 0$. Therefore, particle 2 *gains* kinetic energy. By conservation of energy, the projectile must lose KE, and the separate energies are definitely *not* conserved.

(c) In Fig.14.15 you can see that the final momentum of particle 2 is represented by the line DC, which has magnitude $p'_{\rm lab2} = 2p\sin(\theta_{\rm cm}/2)$. Therefore the energy gained by particle 2 (and lost by particle 1) is

$$\Delta E = \frac{(p'_{\rm lab2})^2}{2m_2} = \frac{2p^2\sin^2(\theta_{\rm cm}/2)}{m_2}.$$

This is to be compared with the original energy of particle 1, which is (in the lab frame)

$$E = T_1 = \frac{(p_{\rm lab1})^2}{2m_1} = \frac{(1+\lambda)^2p^2}{2m_1}$$

Therefore

$$\frac{\Delta E}{E}=\frac{2p^2\sin^2(\theta_{\rm cm}/2)}{m_2}\cdot\frac{2m_1}{(1+\lambda)^2p^2}=\frac{4\lambda\sin^2(\theta_{\rm cm}/2)}{(1+\lambda)^2}.$$

(d) For given λ this fractional loss is greatest if $\theta_{\rm cm}/2 = 90°$ or $\theta_{\rm cm} = 180°$—what one would normally expect in a direct head-on collision. Differentiating with respect to λ, you can easily check that the corresponding fractional loss is maximum if $\lambda = 1$, that is, if the two particles have equal masses.

14.31 ★★★ [Computer] Consider the elastic scattering of a projectile that is heavier than the target, that is, $m_1 > m_2$ or $\lambda > 1$. **(a)** Draw the analog of Figure 14.15 for this case. Show clearly that there are two different values of the CM angle $\theta_{\rm cm}$ corresponding to each value of $\theta_{\rm lab}$. **(b)** What are the two CM angles that correspond to $\theta_{\rm lab} = 0$? In terms of this example, explain why there is this two-fold ambiguity in $\theta_{\rm cm}$ when $m_1 > m_2$. **(c)** Plot $\theta_{\rm lab}$ as a function of $\theta_{\rm cm}$ for the case that $\lambda = 2$. **(d)** Use your picture from part (a) to find an expression for the maximum possible value of $\theta_{\rm lab}$ for a given value of λ. Check that your answer is correct for the case that $\lambda = 1$.

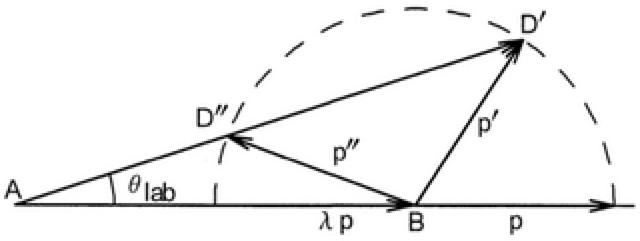

Figure S14.31a

(a) The two final CM momenta labeled $\mathbf{p}'$ and $\mathbf{p}''$ are in different directions, but their corresponding lab momenta, represented by the lines AD' and AD'' are in the same direction.
(b) The lab angle $\theta_{\rm lab}$ is 0 if $\theta_{\rm cm} = 0$ and if $\theta_{\rm cm} = \pi$. In the case $\theta_{\rm cm} = 0$, the projectile passes the target unscattered, and the same is true as seen in the lab; that is, $\theta_{\rm lab} = 0$. In the case $\theta_{\rm cm} = \pi$, the projectile bounces straight back (as seen in the CM frame)—typically in a head-on collision. However, in the lab frame, a head-on collision with $m_1 > m_2$ only slows the projectile, which emerges in the forward direction with $\theta_{\rm lab} = 0$.
(c) See figure S14.31c.

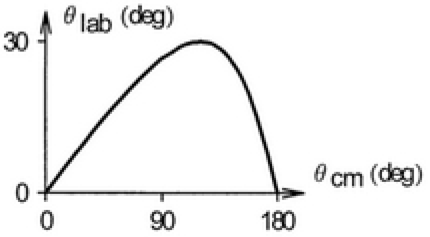

Figure S14.31c

(d) From the figure of part (a), the lab angle $\theta_{\rm lab}$ is maximum when the line $AD''D'$ is tangent to the circle. Therefore $\theta_{\rm lab}(\max) = \arcsin(p/\lambda p) = \arcsin(1/\lambda)$. If $\lambda = 1$, this gives the known result that $\theta_{\rm lab}(\max) = 90°$.

CHAPTER 15

Special Relativity

PROBLEMS AND SOLUTIONS

SECTION 15.2 Galilean Relativity

15.1 ⋆ Using arguments similar to those of Section 15.2, prove that Newton's first and third laws are invariant under the Galilean transformation.

Suppose that the first law holds in a certain frame $\mathcal{S}$. Now consider a second frame $\mathcal{S}'$ obtained from $\mathcal{S}$ by a Galilean transformation and suppose that the net force $\mathbf{F}'$ on a body is zero as measured in $\mathcal{S}'$. By the invariance of force, this means that $\mathbf{F} = \mathbf{0}$ as measured in $\mathcal{S}$. By the first law (true in $\mathcal{S}$), this means that the acceleration $\mathbf{a}$ as measured in $\mathcal{S}$ is zero. Finally, since $\mathbf{v}' = \mathbf{v} - \mathbf{V}$, with $\mathbf{V}$ constant, it follows that $\mathbf{a}' = \mathbf{a} = 0$. In summary, we've proved that

$$(\mathbf{F}' = 0) \quad \Longrightarrow \quad (\mathbf{F} = 0) \quad \Longrightarrow \quad (\mathbf{a} = 0) \quad \Longrightarrow \quad (\mathbf{a}' = 0)$$

and we've proved that the first law holds in $\mathcal{S}'$.

Similarly, we can prove that if the third law holds in $\mathcal{S}$, then

$$\mathbf{F}'_{12} = \mathbf{F}_{12} = -\mathbf{F}_{21} = -\mathbf{F}'_{21}$$

and the third law holds in $\mathcal{S}'$.

SECTION 15.4 The Relativity of Time; Time Dilation

15.3 ⋆ A low-flying earth satellite travels at about 8000 m/s. What is the factor γ for this speed? As observed from the ground, by how much would a clock traveling at this speed differ from a ground-based clock after one hour (as measured by the latter)? What is the percent difference?

If $v = 8000$ m/s, then $\beta = 8000/(3 \times 10^8) = (8/3) \times 10^{-5}$ and

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \approx 1 + \tfrac{1}{2}\beta^2 = 1 + 3.56 \times 10^{-10},$$

where in the third expression I have used the binomial approximation. Since $\Delta t = \gamma \Delta t_o$, the difference is

$$\Delta t_o - \Delta t = (1-\gamma)\Delta t_o \approx -\tfrac{1}{2}\beta^2 \Delta t = -(3.56 \times 10^{-10}) \times 3600\,\text{s} = -1.28\,\mu\text{s}.$$

where in the third expression I have used the binomial approximation for γ and replaced Δt_o by Δt, since the difference is extremely small. The fractional difference is just $-\frac{1}{2}\beta^2 = -3.56 \times 10^{-8}\%$.

15.5 ⋆ A space explorer A sets off at a steady $0.95c$ to a distant star. After exploring the star for a short time, he returns at the same speed and gets home after a total absence of 80 years (as measured by earth-bound observers). How long do A's clocks say that he was gone, and by how much has he aged as compared to his twin B who stayed behind on earth?

[*Note:* This is the famous "twin paradox." It is fairly easy to get the right answer by judicious insertion of a factor of γ in the right place, but to understand it, you need to recognize that it involves *three* inertial frames: the earth-bound frame $\mathcal{S}$, the frame $\mathcal{S}'$ of the outbound rocket, and the frame $\mathcal{S}''$ of the returning rocket. Write down the time dilation formula for the two halves of the journey and then add. Notice that the experiment is *not* symmetrical between the two twins: B stays at rest in the single inertial frame $\mathcal{S}$, but A occupies at least two different frames. This is what allows the result to be unsymmetrical.]

With $\beta = 0.95$, the γ factor for both the outward and return trips is $\gamma = 1/\sqrt{1-\beta^2} = 3.20$. The times for the two halves of the journey satisfy

$$\Delta t_B^{\text{out}} = \gamma \Delta t_A^{\text{out}} \qquad \text{and} \qquad \Delta t_B^{\text{back}} = \gamma \Delta t_A^{\text{back}},$$

so, by addition, the times for the whole journey satisfy the same relation $\Delta t_B = \gamma \Delta t_A$. Therefore $\Delta t_A = \Delta t_B/\gamma = (80\text{ yr})/3.20 = 25$ yr, which is the amount by which twin A has aged.

15.7 ⋆⋆ The muons created by cosmic rays in the upper atmosphere rain down more-or-less uniformly on the earth's surface, although some of them decay on the way down, with a half-life of about 1.5 μs (measured in their rest frame). A muon detector is carried in a balloon to an altitude of 2000 m, and in the course of an hour detects 650 muons traveling at $0.99c$ toward the earth. If an identical detector remains at sea level, how many muons should it register in one hour? Calculate the answer taking account of the relativistic time dilation and also classically. (Remember that after n half-lives, 2^{-n} of the original particles survive.) Needless to say, the relativistic answer agrees with experiment.

With $v = 0.99c$, $\gamma = 7.09$. The half-life (measured in the muons' rest frame) is $t_{1/2}(\text{proper}) = 1.5$ μs, and that measured in the earth frame is $t_{1/2}(\text{earth}) = \gamma t_{1/2}(\text{proper})$. The time of flight measured in the earth frame is $T(\text{earth}) = h/v$, so the number of half-lives that have elapsed is

$$n = \frac{T(\text{earth})}{t_{1/2}(\text{earth})} = \frac{h}{v\gamma t_{1/2}(\text{proper})} = 0.63. \tag{S15.1}$$

Therefore, the number that survive to the ground should be about $N = N_o/2^n = 650/2^{0.63} = 420$.

To find the classical answer, we must delete the factor of γ in the expression (S15.1) for n, to give $n(\text{clas}) = \gamma n(\text{rel}) = 4.49$ half-lives, and $N(\text{clas}) = 650/2^{4.49} \approx 29$.

15.9 ⋆⋆ One way to set up the system of synchronized clocks in a frame $\mathcal{S}$, as described at the beginning of Section 15.4, would be for the chief observer to summon all her helpers to the origin O and synchronize their clocks there, and then have them travel to their assigned positions *very slowly*. Prove this claim as follows: Suppose a certain observer is assigned to a position P at a distance d from the origin. If he travels at constant speed V, when he reaches P how much will his clock differ from the chief's clock at O? Show that this difference approaches 0 as $V \to 0$.

The time for the helper to reach his assigned position (as measured by the chief observer) is $t_c = d/V$. Meanwhile, the time elapsed on the helper's clock is $t_h = t_c/\gamma = d/(V\gamma)$. Thus the difference is

$\delta = t_c - t_h = (d/V)(1 - 1/\gamma)$. We wish to see what happens to this as $V \to 0$, and this requires some care (because of the factor of V in the denominator). However, as $V \to 0$ we can use the binomial expansion to give

$$\delta = \frac{d}{V}\left(1 - \frac{1}{\gamma}\right) = \frac{d}{\beta c}\left(1 - [1 - \beta^2]^{1/2}\right) \approx \frac{d}{\beta c}\left(1 - [1 - \tfrac{1}{2}\beta^2]\right) = \frac{d}{2c}\beta,$$

which approaches zero as $\beta \to 0$.

SECTION 15.5 Length Contraction

15.11⋆ As a meter stick rushes past me (with velocity $\mathbf{v}$ parallel to the stick), I measure its length to be 80 cm. What is v?

The stick's proper length is $l_o = 100$ cm, whereas I measure it to be $l = 80$ cm. Since $l = l_o/\gamma$, we see that $\gamma = 5/4$. Because $\gamma = 1/\sqrt{1-\beta^2}$,

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} = \sqrt{1 - \frac{4^2}{5^2}} = \sqrt{\frac{9}{25}} = \frac{3}{5}.$$

That is, $v = (3/5)c$.

15.13⋆⋆ A meter stick is at rest in frame $\mathcal{S}_o$, which is traveling with speed $V = 0.8c$ in the standard configuration relative to frame $\mathcal{S}$. **(a)** The stick lies in the $x_o y_o$ plane and makes an angle $\theta_o = 60°$ with the x_o axis (as measured in $\mathcal{S}_o$). What is its length l as measured in $\mathcal{S}$, and what is its angle θ with the x axis? [*Hint:* It may help to think of the stick as the hypotenuse of a 30–60–90 triangle of plywood.] **(b)** What is l if $\theta = 60°$? What is θ_o in this case?

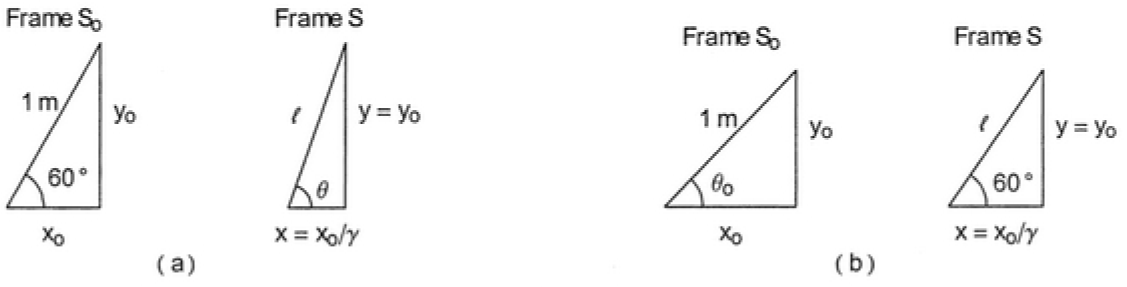

Figure S15.13

(a) With $\beta = 4/5, \gamma = 5/3$. In the frame $\mathcal{S}_o$, we know the length $l_o = 100$ cm and the angle $\theta_o = 60°$, so we can calculate $x_o = 50$ cm and $y_o = 86.6$ cm. In the frame $\mathcal{S}$, x is contracted ($x = x_o/\gamma = 30$ cm) but y is not ($y = y_o = 86.6$ cm). Thence $l = \sqrt{x^2 + y^2} = 91.7$ cm and $\theta = \arctan(y/x) = 70.9°$. **(b)** The angle 60° is given in the frame $\mathcal{S}$, so $\tan 60° = y/x = y_o/(x_o/\gamma)$ and $\tan\theta_o = y_o/x_o = (\tan 60°)/\gamma$, whence $\theta_o = 46.1°$. From this we find $x_o = 69.3$ cm and $y_o = 72.1$ cm, and from these we can calculate $x = x_o/\gamma$ and $y = y_o$ and thence $l = 83.2$ cm.

SECTION 15.6 The Lorentz Transformation

15.15⋆ Solve the Lorentz transformation equations (15.20) to give x, y, z, t in terms of x', y', z', t'. Verify that you get the inverse Lorentz tranformation (15.21). Observe that you could have found the same result by interchanging primed and unprimed variables and changing V to $-V$.

If we multiply the fourth of Eqs. (15.20) by V and add it to the first, we find that

$$x' + Vt' = \gamma x - \gamma V^2 x/c^2 = \gamma(1-\beta^2)x = \sqrt{1-\beta^2}\,x$$

and hence that $x = \gamma(x' + Vt')$. In the same way, we can eliminate x to give $t = \gamma(t' + Vx'/c^2)$. The equations $y = y'$ and $z = z'$ follow trivially, and we have derived the inverse Lorentz transformation (15.21).

15.17 ⋆ Consider two events that occur simultaneously at $t = 0$ in frame $\mathcal{S}$, both on the x axis at $x = 0$ and $x = a$. **(a)** Find the times of the two events as measured in a frame $\mathcal{S}'$ traveling in the positive direction along the x axis with speed V. **(b)** Do the same for a second frame $\mathcal{S}''$ traveling at speed V but in the negative direction along the x axis. Comment on the time ordering of the two events as seen in the three different frames. This startling result is discussed further in Section 15.10.

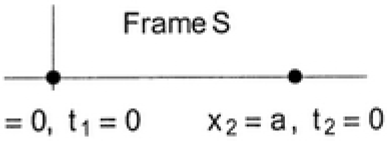

Figure S15.17

(a) Frame $\mathcal{S}'$ has velocity V relative to $\mathcal{S}$. Therefore

$$t_1' = \gamma(t_1 - \beta x_1/c) = 0 \qquad \text{and} \qquad t_2' = \gamma(t_2 - \beta x_2/c) = -\gamma\beta a/c.$$

(b) Frame $\mathcal{S}''$ has velocity $-V$ relative to $\mathcal{S}$. Therefore

$$t_1'' = \gamma(t_1 + \beta x_1/c) = 0 \qquad \text{and} \qquad t_2'' = \gamma(t_2 + \beta x_2/c) = \gamma\beta a/c.$$

In frame $\mathcal{S}$ the two events are simultaneous. In $\mathcal{S}'$ event 1 is later than 2, and in $\mathcal{S}''$ event 1 is earlier than 2.

15.19 ⋆⋆ A traveler in a rocket of proper length $2d$ sets up a coordinate system $\mathcal{S}'$ with its origin O' anchored at the exact middle of the rocket and the x' axis along the rocket's length. At $t' = 0$ she ignites a flashbulb at O'. **(a)** Write down the coordinates x_F', t_F' and x_B', t_B' for the arrival of the light at the front and back of the rocket. **(b)** Now consider the same experiment as observed from a frame $\mathcal{S}$ relative to which the rocket is traveling with speed V (with $\mathcal{S}$ and $\mathcal{S}'$ in the standard configuration). Use the inverse Lorentz transformation to find the coordinates x_F, t_F and x_B, t_B for the arrival of the two signals. Explain clearly in words why the two arrivals are simultaneous in $\mathcal{S}'$ but not in $\mathcal{S}$. This phenomenon is called the relativity of simultaneity.

(a) $x_F' = d,\ t_F' = d/c;\ x_B' = -d,\ t_B' = d/c.$
(b) $x_F = \gamma(x_F' + vt_F') = \gamma(1+\beta)d$ and $t_F = \gamma(t_F' + vx_F'/c^2) = \gamma(1+\beta)d/c$
$x_B = \gamma(x_B' + vt_B') = -\gamma(1-\beta)d$ and $t_B = \gamma(t_B' + vx_B'/c^2) = \gamma(1-\beta)d/c.$

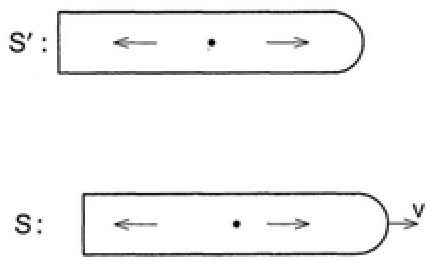

Figure S15.19

Although the two events are simultaneous as measured in S', they are *not* simultaneous in S. As observed in S, the two signals start out from the middle of the rocket, but while they are traveling the rocket is also traveling to the right at speed v. Thus the front is receding from its signal, which must travel more than half the rocket's length. Meanwhile the back of the rocket is approaching its signal, which needs to travel only a shorter distance. Therefore this signal arrives first; that is, $t_B < t_F$. (In S' the signals again start from the middle of the rocket; but since the rocket is not moving they naturally arrive simultaneously.)

SECTION 15.7 The Relativistic Velocity-Addition Formula

15.21 ⋆ A rocket traveling at speed $\frac{1}{2}c$ relative to frame $\mathcal{S}$ shoots forward bullets traveling at speed $\frac{3}{4}c$ relative to the rocket. What is the speed of the bullets relative to $\mathcal{S}$?

Let us take our x axis in the direction of the two velocities. Then the velocity of the rocket's frame $\mathcal{S}'$ has $V = \frac{1}{2}c$ and that of the bullets relative to the rocket has $v'_x = \frac{3}{4}c$, with all other components zero. According to the inverse of the velocity-addition formula (15.26),

$$v_x = \frac{v'_x + V}{1 + v'_x V/c^2} = \frac{\frac{1}{2} + \frac{3}{4}}{1 + \frac{3}{8}}\, c = \frac{5/4}{11/8}\, c = \frac{10}{11}\, c,$$

with all other components zero.

15.23 ⋆ As seen in frame $\mathcal{S}$, two rockets are approaching one another along the x axis traveling with equal and opposite velocities of $0.9c$. What is the velocity of the rocket on the right as measured by observers in the one on the left? [This and the previous two problems illustrate the general result that in relativity the "sum" of two velocities that are less than c is always less than c. See Problem 15.43.]

If we let $\mathcal{S}'$ denote the rest frame of the left rocket, then the velocity of $\mathcal{S}'$ relative to $\mathcal{S}$ is $V = 0.9c$. The velocity of the right rocket relative to $\mathcal{S}$ has $v_x = -0.9c$ and, relative to $\mathcal{S}'$,

$$v'_x = \frac{v_x - V}{1 - v_x V/c^2} = \frac{-0.9c - 0.9c}{1 + 0.9 \times 0.9} = -\frac{1.8c}{1.81} = -0.994c.$$

15.25 ⋆ A rocket is traveling at speed V along the x axis of frame $\mathcal{S}$. It emits a signal (for example, a pulse of light) that travels with speed c along the y' axis of the rocket's rest frame $\mathcal{S}'$. What is the speed of the signal as measured in $\mathcal{S}$?

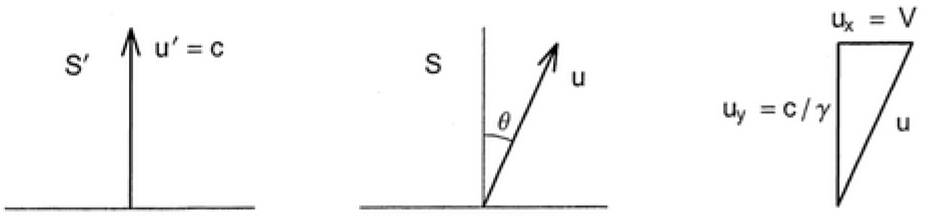

Figure S15.25

According to the inverse velocity transformation,

$$u_x = \frac{u'_x + V}{1 + u'_x V/c^2} = V, \quad u_y = \frac{u'_y}{\gamma(1 + u'_x V/c^2)} = c/\gamma, \quad u_z = 0$$

As you would expect, the signal angles to the right of the y axis, as seen in $\mathcal{S}$. Its speed is given by

$$u^2 = u_x^2 + u_y^2 = V^2 + c^2/\gamma^2 = V^2 + c^2(1 - V^2/c^2) = c^2.$$

That is, $u = c$. This illustrates the general result that anything which has speed c in one frame has the same speed in *any* frame.

15.27 ⋆⋆⋆ Frame $\mathcal{S}'$ travels at speed V_1 along the x axis of frame $\mathcal{S}$ (in the standard configuration). Frame $\mathcal{S}''$ travels at speed V_2 along the x' axis of frame $\mathcal{S}'$ (also in the standard configuration). By applying the standard Lorentz transformation twice find the coordinates x'', y'', z'', t'' of any event in terms of x, y, z, t. Show that this transformation is in fact the standard Lorentz transformation with velocity V given by the relativistic "sum" of V_1 and V_2.

Consider first the final x coordinate. This is $x'' = \gamma_2(x' - V_2 t')$, where γ_2 is the γ factor corresponding to the second velocity V_2. The coordinates x' and t' are given by the first Lorentz transformation (velocity V_1), and substituting these values we find that

$$\begin{aligned} x'' &= \gamma_1\gamma_2[(x - V_1 t) - V_2(t - V_1 x/c^2)] \\ &= \gamma_1\gamma_2[x(1 + V_1V_2/c^2) - (V_1 + V_2)t] \\ &= \gamma_1\gamma_2\left(1 + \frac{V_1V_2}{c^2}\right)\left[x - \left(\frac{V_1 + V_2}{1 + V_1V_2/c^2}\right)t\right] \\ &= \gamma_1\gamma_2(1 + \beta_1\beta_2)[x - Vt] \end{aligned} \tag{S15.2}$$

In the last line I have made two substitutions. In the interests of tidiness, I have replaced V_1/c by β_1 and V_2/c by β_2. More important, I have recognized that the coefficient of t in the previous line is the relativistic "sum"

$$V = \frac{V_1 + V_2}{1 + V_1V_2/c^2}$$

of the two separate velocities V_1 and V_2. The form (S15.2) for x'' is very close to the standard Lorentz transformation for the single velocity V. All that remains to be shown is that the product to the left of the square bracket is equal to the γ factor for the velocity V. To show this, let's evaluate the latter:

$$\gamma_V = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-(\beta_1+\beta_2)^2/(1+\beta_1\beta_2)^2}} = \frac{(1+\beta_1\beta_2)}{\sqrt{(1+\beta_1\beta_2)^2-(\beta_1+\beta_2)^2}}$$

$$= \frac{(1+\beta_1\beta_2)}{\sqrt{(1-\beta_1^2)(1-\beta_2^2)}}.$$

Comparing with (S15.2) we see that indeed $x'' = \gamma_V(x - Vt)$; that is, the two successive Lorentz transformations with velocities V_1 and V_2 produce the same effect as a single Lorentz transformation with velocity V equal to the relativistic "sum" of V_1 and V_2. The transformations of y and z are trivial (for example, $y'' = y' = y$) and that of the time works just the same as that of x.

SECTION 15.8 Four-Dimensional Space–Time; Four-Vectors

15.29 ⋆ **(a)** Find the 3 × 3 matrix $\mathbf{R}(\theta)$ that rotates three-dimensional space about the x_3 axis, so that $\mathbf{e}_1$ rotates through angle θ toward $\mathbf{e}_2$. **(b)** Show that $[\mathbf{R}(\theta)]^2 = \mathbf{R}(2\theta)$, and interpret this result.

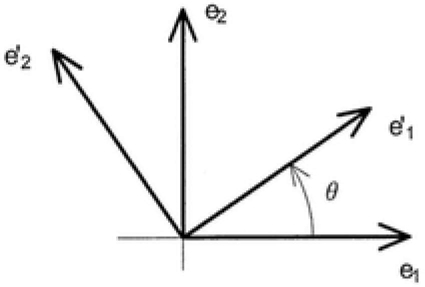

Figure S15.29

(a) Bearing in mind that $R_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j$, we can read off the elements of $\mathbf{R}(\theta)$ from the picture:

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Using the standard rules of matrix multiplication to multiply $\mathbf{R}(\theta)$ by itself, we find

$$[\mathbf{R}(\theta)]^2 = \begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\cos\theta\sin\theta & 0 \\ -2\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta & 0 \\ -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{R}(2\theta).$$

This expresses the obvious result that two successive rotations of θ about a given axis produce the same effect as a single rotation of 2θ about the same axis.

15.30 ⋆ The "angle" ϕ introduced in connection with Equation (15.40) has several useful properties. For any speed $v < c$ (with corresponding factors β and γ) we can define ϕ so that $\gamma = \cosh\phi$. Defined in this way, ϕ is called the **rapidity** corresponding to v. Prove that $\sinh\phi = \beta\gamma$ and that $\tanh\phi = \beta$.

15.31 ⋆ Here is a handy property of the rapidity introduced in Problem 15.30: Suppose that observer B has rapidity ϕ_1 as measured by A and that C has rapidity ϕ_2 as measured by B (with both velocities along the x axis). That is, the speed of B relative to A has $\beta_1 = \tanh\phi_1$ and so on. Prove that the rapidity of C as measured by A is just $\phi = \phi_1 + \phi_2$.

The speed $v = \beta c$ of C relative to A is given by the inverse velocity addition formula:

$$\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2} = \frac{\tanh\phi_1 + \tanh\phi_2}{1 + (\tanh\phi_1)(\tanh\phi_2)} = \tanh(\phi_1 + \phi_2)$$

(In the last step I used the "well known" addition formula for the hyperbolic tangent. One way to prove this is to start from the corresponding formula for the more familiar trigonometric tangent.) That is, the velocity of C relative to A is given by $\beta = \tanh\phi$ where the rapidity ϕ is just $\phi = \phi_1 + \phi_2$.

15.33 ⋆⋆ **(a)** By exchanging x_1 and x_2, write down the Lorentz transformation for a boost of velocity V along the x_2 axis and the corresponding 4×4 matrix Λ_{B2}. **(b)** Write down the 4×4 matrices $\Lambda_{\mathrm{R+}}$ and $\Lambda_{\mathrm{R-}}$ that represent rotations of the x_1x_2 plane through $\pm\pi/2$, with the angle of rotation measured counterclockwise. **(c)** Verify that $\Lambda_{\mathrm{B2}} = \Lambda_{\mathrm{R-}}\Lambda_{\mathrm{B1}}\Lambda_{\mathrm{R+}}$, where Λ_{B1} is the standard boost along the x_1 axis, and interpret this result.

(a)

$$\left.\begin{aligned} x_1' &= x_1 \\ x_2' &= \gamma(x_2 - \beta x_4) \\ x_3' &= x_3 \\ x_4' &= \gamma(x_4 - \beta x_2) \end{aligned}\right\} \quad \text{whence} \quad \Lambda_{\mathrm{B2}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & 0 & -\gamma\beta \\ 0 & 0 & 1 & 0 \\ 0 & -\gamma\beta & 0 & \gamma \end{bmatrix}$$

(b)

$$\Lambda_{\mathrm{R+}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \Lambda_{\mathrm{R-}} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c)

$$\begin{aligned} \Lambda_{\mathrm{R-}}\Lambda_{\mathrm{B1}}\Lambda_{\mathrm{R+}} &= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \gamma & 0 & -\gamma\beta \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\gamma\beta & 0 & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & 0 & -\gamma\beta \\ 0 & 0 & 1 & 0 \\ 0 & -\gamma\beta & 0 & \gamma \end{bmatrix} = \Lambda_{\mathrm{B2}} \end{aligned}$$

The rotation $\Lambda_{\mathrm{R+}}$ changes what was the y axis into the new x axis. Then the boost Λ_{B1} boosts to velocity V along the new x axis, and finally, the rotation $\Lambda_{\mathrm{R-}}$ converts the new x axis back to being the y axis. The net effect of all three is therefore a boost along the y axis.

15.35 ⋆⋆ Prove the following useful result, called the *zero-component theorem*: Let q be a four-vector, and suppose that one component of q is found to be zero in all inertial frames. (For example, $q_4 = 0$ in all frames.) Then all four components of q are zero in all frames.

Suppose that $q_4 = 0$ in all inertial frames. Now consider any frame $\mathcal{S}$ and a second one $\mathcal{S}'$ related to $\mathcal{S}$ by the standard boost, so that $q_4' = \gamma(q_4 - \beta q_1)$. Since $q_4 = q_4' = 0$, it follows that $q_1 = 0$. But the frame $\mathcal{S}$ was arbitrary. Therefore $q_1 = 0$ in all frames. If we repeat the argument using boosts along the y and z axes, we can prove similarly that q_2 and q_3 are zero in all frames.

SECTION 15.9 The Invariant Scalar Product

15.37 ⋆ Verify directly that $x' \cdot y' = x \cdot y$ for any two four-vectors x and y, where x' and y' are related to x and y by the standard Lorentz boost along the x_1 axis.

If x and y are four-vectors, then under the standard boost $x'_1 = \gamma(x_1 - \beta x_4)$, $x'_2 = x_2$, and so on. Therefore

$$\begin{aligned} x' \cdot y' &= x'_1 y'_1 + x'_2 y'_2 + x'_3 y'_3 - x'_4 y'_4 \\ &= \gamma(x_1 - \beta x_4)\gamma(y_1 - \beta y_4) + x_2 y_2 + x_3 y_3 - \gamma(x_4 - \beta x_1)\gamma(y_4 - \beta y_1) \\ &= \gamma^2(1 - \beta^2) x_1 y_1 + x_2 y_2 + x_3 y_3 - \gamma^2(1 - \beta^2) x_4 y_4 \end{aligned}$$

since the cross terms involving $x_1 y_4$ and $x_4 y_1$ in the second line all cancelled. Finally, notice that $\gamma^2(1 - \beta^2) = 1$, so the last line is just $x \cdot y$, and we've proved that $x' \cdot y' = x \cdot y$.

SECTION 15.10 The Light Cone

15.39 ⋆ Suppose that a point P in space–time with coordinates $x = (\mathbf{x}, x_4)$ lies inside the backward light cone as seen in frame $\mathcal{S}$. This means that $x \cdot x < 0$ and $x_4 < 0$ at least in frame $\mathcal{S}$. Prove that these two conditions are satisfied in all frames. Since this means that all observers agree that $t < 0$, this justifies calling the inside of the backward light cone the absolute past.

If $x \cdot x < 0$ in frame $\mathcal{S}$, then $x' \cdot x' < 0$ in any other frame $\mathcal{S}'$, since $x \cdot x$ has the same value in all frames. The condition $x \cdot x = \mathbf{x}^2 - x_4^2 < 0$ implies that $|\mathbf{x}| < |x_4|$. Now suppose, in addition, that $x_4 < 0$ in frame $\mathcal{S}$ and let $\mathcal{S}'$ be obtained from $\mathcal{S}$ by a standard boost. Then

$$x'_4 = \gamma(x_4 - \beta x_1) < 0$$

since $|\beta| < 1$ and $|x_1| \le |\mathbf{x}| < |x_4|$. That is, $x'_4 < 0$ in $\mathcal{S}'$. Since x_4 is unchanged by any rotation, the same conclusion holds in all frames $\mathcal{S}'$.

15.41 ⋆ In the proposition on page 627, it is obvious that at least one of the three statements has to be true. In the proof given there, I showed that if statement (1) is true, then so are statements (2) and (3). To complete the proof, show that (2) implies (1) and (3). [Strictly speaking you should also check that (3) implies (1) or (2), but this is so similar to the argument already given that you needn't bother.]

To simplify our notation, I'll put the origin at point Q. Then statement (2) says that the point P has coordinates x satisfying $x^2 > 0$ and that there is a frame (call it $\mathcal{S}$) in which $x_4 = 0$. We have to prove that there are other frames where $x'_4 < 0$ and still others where $x''_4 > 0$. By rotating our coordinates, if necessary, we can arrange that $\mathbf{x}$ lies on the positive x_1 axis and $x = (x_1, 0, 0, 0)$, with $x_1 > 0$. Now consider a frame $\mathcal{S}'$ obtained from $\mathcal{S}$ by a standard boost. In this frame

$$x'_4 = \gamma(x_4 - \beta x_1) = -\gamma\beta x_1 < 0.$$

Similarly, by boosting in the opposite direction we could obtain a frame $\mathcal{S}''$ in which $x''_4 = +\gamma\beta x_1 > 0$.

15.43 ⋆ **(a)** Show that if a body has speed $v < c$ in one inertial frame, then $v < c$ in all frames. [*Hint:* Consider the displacement four-vector $dx = (d\mathbf{x}, c\,dt)$, where $d\mathbf{x}$ is the three-dimensional displacement in a short time dt.] **(b)** Show similarly that if a signal (such as a pulse of light) has speed c in one frame, its speed is c in all frames.

(a) Suppose that the body moves from $\mathbf{x}$ to $\mathbf{x} + d\mathbf{x}$ as the time advances from t to $t + dt$. Let dx denote the four-vector displacement $dx = (d\mathbf{x}, c\,dt) = (\mathbf{v}, c)dt$ and consider the following two equivalent statements:

$$|\mathbf{v}| < c \iff dx^2 < 0. \tag{S15.3}$$

Since dx^2 is Lorentz invariant, the second condition, if true in one frame, must be true in all frames. The same must therefore apply to the first; that is, if $|\mathbf{v}| < c$ in one frame, then $|\mathbf{v}| < c$ in all frames. **(b)** The argument for a signal with speed c is the same except that the two conditions (S15.3) are replaced by $|\mathbf{v}| = c \iff dx^2 = 0$.

SECTION 15.11 The Quotient Rule and Doppler Effect

15.45 ⋆ The quotient rule derived at the start of Section 15.11 is only one of several similar quotient rules. Here is another. Suppose that k and x are both known to be four-vectors and that in every inertial frame k is a multiple of x. That is, $k = \lambda x$ in frame $\mathcal{S}$, and $k' = \lambda' x'$ in frame $\mathcal{S}'$, and so on. Then the factor λ (the "quotient" of k and x) is in fact a four-scalar with the same value in all frames, $\lambda = \lambda'$. Prove this quotient rule.

If k and x are both four-vectors, then for any two frames $\mathcal{S}$ and $\mathcal{S}'$, $k' = \Lambda k$ and $x' = \Lambda x$. Now since $k = \lambda x$,

$$\Lambda k = \Lambda \lambda x = \lambda \Lambda x = \lambda x'.$$

On the other hand, $\Lambda k = k' = \lambda' x'$. Comparing these two expressions for Λk, we conclude that $\lambda = \lambda'$ and we've proved that λ is a four-scalar. (The only exception is if all four components of x' are zero, but this case is of little interest since then $x = k = 0$.)

15.47 ⋆ Consider the tale of the physicist who is ticketed for running a red light and argues that because he was approaching the intersection, the red light was Doppler shifted and appeared green. How fast would he have to have been going? ($\lambda_{\text{red}} \approx 650$ nm and $\lambda_{\text{green}} \approx 530$ nm.)

Since he had to be approaching the light head-on (or nearly so), $\theta = 0$ and Eq. (15.64) becomes $\omega = \omega_o\sqrt{(1+\beta)/(1-\beta)}$ (as in Problem 1.46). Solving for β we find that

$$\beta = \frac{\omega^2 - \omega_o^2}{\omega^2 + \omega_o^2} = \frac{\lambda_o^2 - \lambda^2}{\lambda_o^2 + \lambda^2} = \frac{65^2 - 53^2}{65^2 + 53^2} = 0.20.$$

His speed had to be about $0.20c$.

SECTION 15.12 Mass, Four-Velocity, and Four-Momentum

15.49 ⋆ Show that the four-velocity of any object has invariant length squared $u \cdot u = -c^2$.

Since $u = \gamma(\mathbf{v}, c)$, its square is $u^2 = \gamma^2(v^2 - c^2) = \dfrac{1}{1-\beta^2}(\beta^2 - 1)c^2 = -c^2$.

15.51 ⋆⋆ **(a)** For the collision shown in Figure 15.11, verify that all four components of the total four-momentum $p_a + p_b$ [with the individual momenta defined relativistically as in (15.68)] are conserved in the frame $\mathcal{S}$ of part (a). **(b)** In two lines or less, prove that total four-momentum is conserved in the frame $\mathcal{S}'$ of part (b). [This problem does not, of course, prove that the law of conservation of four-momentum is generally true, but it does at least show that the law is consistent with the collision of Figure 15.11.]

(a) The two incoming particles have the same mass, $m_a = m_b = m$. Since their velocities are equal and opposite, $\mathbf{v}_a = -\mathbf{v}_b = \mathbf{v}$, say, they have the same value of γ, that is, $\gamma_a = \gamma_b = \gamma$, say. It follows that $p_a = (\gamma m\mathbf{v}, \gamma mc)$ and $p_b = (-\gamma m\mathbf{v}, \gamma mc)$. Therefore the total initial four-momentum is $\sum p_{\text{in}} = (\mathbf{0}, 2\gamma mc)$. The same argument gives the same value for the final total four-momentum, and we've shown that $\sum p_{\text{in}} = \sum p_{\text{fin}}$ in frame $\mathcal{S}$.
(b) Since the two sides of the last equation are four-vectors, the truth of the equation in one inertial frame automatically assures its truth in all such frames.

15.53 ⋆⋆ For any two objects a and b, show that

$$p_a \cdot p_b = -m_a E_b = -m_b E_a = -m_a m_b c^2 \gamma(v_{\text{rel}})$$

where m_a is the mass of a, and E_b is the energy of b in a's rest frame, and vice versa, and v_{rel} is the speed of a in the rest frame of b (or vice versa).

The quantity $p_a \cdot p_b$ is invariant, so can be evaluated in any conveniently chosen frame. In the rest frame of a, $p_a = (\mathbf{0}, m_a c)$ and $p_b = (\mathbf{p}_b, E_b/c)$, so $p_a \cdot p_b = -m_a E_b$. Still in the same frame, the speed of b is v_{rel}, so its energy is $E_b = \gamma(v_{\text{rel}}) m_b c^2$, and $p_a \cdot p_b = -m_a m_b c^2 \gamma(v_{\text{rel}})$. Finally, working in the same way, but in the rest frame of b, we find that $p_a \cdot p_b = -m_b E_a$.

15.55 ⋆⋆⋆ Since the four-velocity $u = \gamma(\mathbf{v}, c)$ is a four-vector its transformation properties are simple. Write down the standard Lorentz boost for all four components of u. Use these to deduce the relativistic velocity-addition formula for $\mathbf{v}$.

Because there are three different velocities in this problem, we must be careful to distinguish the different corresponding factors of β and γ. Thus I'll write $u = \gamma(v)(\mathbf{v}, c)$, and the velocity transformation as

$$u_1' = \gamma(V)(u_1 - \beta(V)u_4), \quad u_2' = u_2, \quad u_3' = u_3, \quad u_4' = \gamma(V)(u_4 - \beta(V)u_4). \tag{S15.4}$$

We must now rewrite these in terms of the three-velocity $\mathbf{v}$. The first gives

$$\gamma(v')v_1' = \gamma(V)[\gamma(v)v_1 - \beta(V)\gamma(v)c] = \gamma(v)\gamma(V)[v_1 - V] \tag{S15.5}$$

and the last gives

$$\gamma(v')c = \gamma(V)[\gamma(v)c - \beta(V)\gamma(v)v_1] = \gamma(v)\gamma(V)c[1 - v_1V/c^2]. \tag{S15.6}$$

Dividing Eq. (S15.5) by Eq. (S15.6), we find

$$v_1' = \frac{v_1 - V}{1 - v_1V/c^2}$$

which is the first component of the velocity-addition formula. Similarly, the second of Eqs. (S15.4) gives $\gamma(v')v_2' = \gamma(v)v_2$ and, dividing this by Eq. (S15.6), we find

$$v_2' = \frac{v_2}{\gamma(V)(1 - v_1 V/c^2)}$$

which is the second component. The third works in the same way, and we're home.

SECTION 15.13 Energy, the Fourth Component of Momentum

15.57 ⋆ When a radioactive nucleus of astatine 215 decays at rest, the whole atom is torn into two in the reaction

$$^{215}\text{At} \to \, ^{211}\text{Bi} + \, ^{4}\text{He}.$$

The masses of the three atoms are (in order) 214.9986, 210.9873, and 4.0026, all in atomic mass units. (1 atomic mass unit = 1.66×10^{-27} kg = 931.5 MeV/c^2.) What is the total kinetic energy of the two outcoming atoms, in joules and in MeV?

With the initial atom at rest, conservation of energy implies that $M_\text{i}c^2 = M_\text{f}c^2 + T_\text{f}$, so $T_\text{f} = (M_\text{i} - M_\text{f})c^2 = (m_\text{At} - m_\text{Bi} - m_\text{He})c^2 = (0.0087\ \text{u})c^2 = 8.1$ MeV = 1.3×10^{-12} J.

15.59 ⋆ If one defines a variable mass $m_\text{var} = \gamma m$, then the relativistic momentum $\mathbf{p} = \gamma m\mathbf{v}$ becomes $m_\text{var}\mathbf{v}$ which looks more like the classical definition. Show, however, that the relativistic kinetic energy is *not* equal to $\frac{1}{2}m_\text{var}v^2$.

The correct relativistic KE is $T = E - mc^2 = (\gamma - 1)mc^2$. The question is: Could this be the same as $\frac{1}{2}m_\text{var}v^2 = \frac{1}{2}\gamma m v^2$? If these were the same, then it would have to be that $(\gamma - 1)c^2 = \frac{1}{2}\gamma v^2$ or $\gamma = 1/(1 - \frac{1}{2}v^2/c^2)$ which is certainly false.

15.61 ⋆ A particle of mass 3 MeV/c^2 has momentum 4 MeV/c. What are its energy (in MeV) and speed (in units of c)?

$E = \sqrt{(pc)^2 + (mc^2)^2} = \sqrt{(4\ \text{MeV})^2 + (3\ \text{MeV})^2} = 5\ \text{MeV}$.
$\beta = pc/E$ = (4 MeV)/(5 MeV) = 0.8. Therefore $v = 0.8c$.

15.63 ⋆ **(a)** What is a mass of 1 MeV/c^2 in kilograms? **(b)** What is a momentum of 1 MeV/c in kg·m/s?

(a) $1\,\dfrac{\text{MeV}}{c^2} = \dfrac{1.60 \times 10^{-13}\ \text{J}}{(3.00 \times 10^8\ \text{m/s})^2} = 1.78 \times 10^{-30}$ kg.
(b) $1\,\dfrac{\text{MeV}}{c} = \dfrac{1.60 \times 10^{-13}\ \text{J}}{3.00 \times 10^8\ \text{m/s}} = 5.33 \times 10^{-22}$ kg·m/s. (The final 3 here contains a rounding error and should really be a 4.)

15.65 ⋆⋆ The relativistic kinetic energy of a particle is $T = (\gamma - 1)mc^2$. Use the binomial series to express T as a series in powers of $\beta = v/c$. **(a)** Verify that the first term is just the nonrelativistic kinetic energy, and show that to lowest order in β the difference between the relativistic and nonrelativistic kinetic energies is $3\beta^4 mc^2/8$. **(b)** Use this result to find the maximum speed at which the nonrelativistic value is within 1% of the correct relativistic value.

$$\begin{aligned} T &= mc^2(\gamma - 1) = mc^2[(1-\beta^2)^{-1/2} - 1] \\ &= mc^2\left[\left(1 + \tfrac{1}{2}\beta^2 + \tfrac{3}{8}\beta^4 + \tfrac{5}{16}\beta^6 + \cdots\right) - 1\right] \\ &= mc^2\left[\tfrac{1}{2}\beta^2 + \tfrac{3}{8}\beta^4 + \tfrac{5}{16}\beta^6 + \cdots\right] \end{aligned}$$

.

(a) The first term is $\frac{1}{2}mc^2\beta^2 = \frac{1}{2}mv^2 = T_{\text{nr}}$. The difference is $T_{\text{rel}} - T_{\text{nr}} = mc^2[\frac{3}{8}\beta^4 + \cdots]$.
(b) The fractional difference is $\dfrac{T_{\text{rel}} - T_{\text{nr}}}{T_{\text{rel}}} \approx \dfrac{\frac{3}{8}\beta^4}{\frac{1}{2}\beta^2} = \frac{3}{4}\beta^2$, and this is less than 1% as long as $\beta < 0.12$.

SECTION 15.14 Collisions

15.67 ⋆ Two balls of equal masses (m each) approach one another head-on with equal but opposite velocities of magnitude $0.8c$. Their collision is perfectly inelastic, so they stick together and form a single body of mass M. What is the velocity of the final body and what is its mass M?

With $v = 0.8c$, $\gamma = 5/3$. Since the initial velocities are equal and opposite, the same is true of the momenta. Therefore the total momentum is zero, and the final body is at rest. Thus $M = E_{\text{fin}}/c^2 = E_{\text{in}}/c^2 = 2\gamma m = 3.33m$.

15.69 ⋆ **(a)** Show that the four-momentum of any material particle ($m > 0$) is forward time-like. **(b)** Show that the sum of any two forward time-like vectors is itself forward time-like, and hence that the sum of any number of forward time-like vectors is itself forward time-like.

(a) A four-vector q is forward time-like if and only if $|\mathbf{q}| < q_4$. The four-momentum of a massive particle is defined as $p = mu = \gamma m(\mathbf{v}, c)$, and, since $m > 0$, $|\mathbf{v}| < c$ and hence $|\mathbf{p}| < p_4$. Therefore p is forward time-like.
(b) If p and q are forward time-like, $|\mathbf{p}| < p_4$ and $|\mathbf{q}| < q_4$. It follows that $|\mathbf{p} + \mathbf{q}| \le |\mathbf{p}| + |\mathbf{q}| < p_4 + q_4 = (p+q)_4$. Therefore, $p + q$ is forward time-like.

15.71 ⋆ One way to create exotic heavy particles is to arrange a collision between two lighter particles

$$a + b \to d + e + \cdots + g$$

where d is the heavy particle of interest and $e, \cdots, g$ are other possible particles produced in the reaction. (A good example of such a process is the production of the ψ particle in the process $e^+ + e^- \to \psi$, in which there are no other particles $e, \cdots, g$.) **(a)** Assuming that m_d is much heavier that any of the other particles, show that the minimum (or threshold) energy to produce this reaction in the CM frame is $E_{\text{cm}} \approx m_d c^2$. **(b)** Show that the threshold energy to produce the same reaction in the lab frame, where the particle b is initially at rest, is $E_{\text{lab}} \approx m_d^2 c^2/2m_b$. **(c)** Calculate these two energies for the process $e^+ + e^- \to \psi$, with $m_e \approx 0.5$ MeV/c^2 and $m_\psi \approx 3100$ MeV/c^2. Your answers should explain why particle physicists go to the trouble and expense of building colliding-beam experiments.

(a) The total energy needed to produce any given final set of particles must satisfy $E \geq \sum m_{\text{fin}}c^2$. The unique feature of the CM frame is that with $\mathbf{p}_{\text{tot}} = 0$, the final particles can all be at rest and the inequality can actually be an equality. Thus the threshold energy in the CM frame is $\sum m_{\text{fin}}c^2$. If the particle d is much heavier than all the others, this gives $E_{\text{cm}} \approx m_d c^2$ for the threshold.
(b) In the lab frame, with particle b at rest initially, the threshold energy is given by (15.98) as

$$E_{\text{lab}} = \frac{\left(\sum m_{\text{fin}}\right)^2 - m_a^2 - m_b^2}{2m_b}c^2 \approx \frac{m_d^2 c^2}{2m_b}$$

where the final expression holds if particle d is much heavier than all the others. (Strictly speaking this is the minimum energy for the incident particle a—as opposed to the minimum total energy—but if m_d is much bigger than all other masses, this difference is unimportant.)
(c) Putting in $m_b = m_e = 0.5\ \text{MeV}/c^2$ and $m_d = m_\psi = 3100\ \text{MeV}/c^2$, we find for the two threshold energies

$$E_{\text{cm}} \approx m_\psi c^2 = 3100\ \text{MeV} \quad \text{whereas} \quad E_{\text{lab}} \approx \frac{m_\psi^2 c^2}{2m_e} = 9,600,000\ \text{MeV}.$$

15.73 ⋆⋆ Consider the head-on elastic collision of Example 15.10, in which the final velocity $\mathbf{v}_b$ of particle b is given by (15.95). **(a)** Show that, in the special case that the masses are equal ($m_a = m_b$), $v_b = v_a$, the initial velocity of particle a. Show that in this case the final velocity of a is zero. [This result for equal-mass collisions is well known in classical mechanics; you have now shown that it extends to relativity.] **(b)** Show that in the nonrelativistic limit (15.95) reduces to $v_b = 2v_a m_a/(m_a + m_b)$. By doing the necessary nonrelativistic calculations, show that this agrees with the nonrelativistic answer for elastic head-on collisions.

(a) According to (15.95), the final velocity of b is $\mathbf{v}_b = 2\boldsymbol{\beta}c/(1+\beta^2)$ where $\boldsymbol{\beta}$ is the (dimensionless) velocity of the CM frame relative to the lab, $\boldsymbol{\beta} = \mathbf{p}_a c/(E_a + mc^2)$ and $m = m_a = m_b$. Putting these together, we find, after a little algebra, $\mathbf{v}_b = \mathbf{p}_a c^2/E_a = \mathbf{v}_a$.
(b) In the limit that $\beta \to 0$,

$$\mathbf{v}_b = \frac{2\boldsymbol{\beta}c}{1+\beta^2} \to 2\boldsymbol{\beta}c = \frac{2\mathbf{p}_a c^2}{E_a + m_b c^2} \to \frac{2m_a \mathbf{v}_a}{m_a + m_b}.$$

As you can easily check, this is the answer you would get using conservation of nonrelativistic energy and momentum.

15.75 ⋆⋆ A particle of unknown mass M decays into two particles of known masses $m_a = 0.5\ \text{GeV}/c^2$ and $m_b = 1.0\ \text{GeV}/c^2$, whose momenta are measured to be $\mathbf{p}_a = 2.0\ \text{GeV}/c$ along the x_2 axis and $\mathbf{p}_b = 1.5\ \text{GeV}/c$ along the x_1 axis. (1 GeV = 10^9 eV.) Find the unknown mass M and its speed.

Using the useful relation (15.85) we can find the energies of the two final particles

$$E_1 = \sqrt{(p_1 c)^2 + (m_1 c^2)^2} = \sqrt{2^2 + (0.5)^2} = 2.06\ \text{GeV}$$

and

$$E_2 = \sqrt{(p_2 c)^2 + (m_2 c^2)^2} = \sqrt{(1.5)^2 + 1^2} = 1.80\ \text{GeV}$$

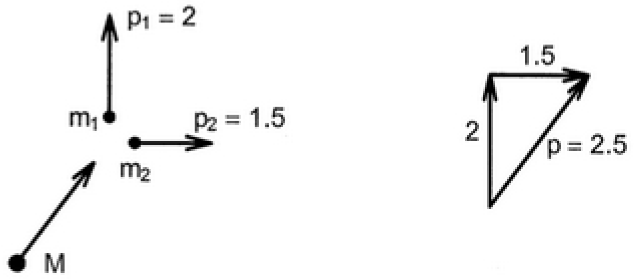

Figure S15.75

By conservation of energy and momentum, the original particle had

$$E = E_1 + E_2 = 3.86 \text{ GeV}$$

and

$$\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 = 2.5 \,\text{GeV}/c, \text{ in the direction shown.}$$

Finally, $M = \sqrt{E^2 - (pc)^2} = \sqrt{3.86^2 - 2.5^2} = 2.95 \text{ GeV}/c^2$, and $\beta = pc/E = 2.5/3.86 = 0.65$.

15.77 ★★★ Consider the elastic head-on collision of Example 15.10, in which particle a collides with a stationary particle b. Assuming that $m_a \neq m_b$, show that the final kinetic energy of particle a satisfies $T_a^{\text{fin}} < (m_a - m_b)^2c^2/2m_b$. [*Hint:* Look at the CM frame where you can show that the four-vector $p_a^{\text{fin}} - p_b^{\text{in}}$ is time-like, so that $(p_a^{\text{fin}} - p_b^{\text{in}})^2 < 0$.] **(b)** The result of part (a) implies that if T_a^{in} is large, almost all this incoming energy is lost to b. This is quite different from the nonrelativistic situation. Prove that in nonrelativistic mechanics the proportion of kinetic energy retained by a is fixed, independent of T_a^{in}. Specifically, $T_a^{\text{fin}} = T_a^{\text{in}}(m_a - m_b)^2/(m_a + m_b)^2$.

(a) In the CM frame we know that the two incoming momenta are equal and opposite and that, in a head-on collision, they simply exchange roles. Therefore $\mathbf{p}_a^{\text{fin}} = \mathbf{p}_b^{\text{in}}$, and $p_a^{\text{fin}} - p_b^{\text{in}} = (\mathbf{0}, E_a^{\text{fin}} - E_b^{\text{in}})/c$, which is pure time-like. Therefore $(p_a^{\text{fin}} - p_b^{\text{in}})^2 < 0$. (The one exception is if $m_a = m_b$, in which case $E_a^{\text{fin}} = E_b^{\text{in}}$, so $p_a^{\text{fin}} - p_b^{\text{in}} = 0$.)

Now, in the lab frame (particle b initially at rest) $p_b^{\text{in}} = (\mathbf{0}, m_b c)$, so the condition $(p_a^{\text{fin}} - p_b^{\text{in}})^2 < 0$ can be written as

$$-m_a^2c^2 - m_b^2c^2 - 2p_a^{\text{fin}} \cdot p_b^{\text{in}} = -m_a^2c^2 - m_b^2c^2 + 2E_a^{\text{fin}}m_b < 0.$$

If we write $E_a^{\text{fin}} = m_a c^2 + T_a^{\text{fin}}$, this gives the desired result, $T_a^{\text{fin}} < (m_a - m_b)^2c^2/2m_b$. (Notice that if $m_a = m_b$, the corresponding result is that $T_a^{\text{fin}} = 0$, giving a nice proof that, in this case, particle a comes to a dead stop.)

(b) Using nonrelativistic mechanics, it's easy to show that $\mathbf{v}_a^{\text{fin}} = \mathbf{v}_a^{\text{in}}(m_a - m_b)/(m_a + m_b)$, whence $T_a^{\text{fin}} = T_a^{\text{in}}(m_a - m_b)^2/(m_a + m_b)^2$.

SECTION 15.15 Force in Relativity

15.79 ★ Consider an object of mass m (which you may assume is constant), acted on by a force $\mathbf{F}$. From the definition (15.100) prove that

$$\mathbf{F} = \gamma m\mathbf{a} + (\mathbf{F} \cdot \mathbf{v})\mathbf{v}/c^2,$$

where $\mathbf{a} = d\mathbf{v}/dt$ is the object's acceleration. Notice that it is certainly not true in relativity that $\mathbf{F} = m\mathbf{a}$. Nor is it true that $\mathbf{F} = m_{\text{var}}\mathbf{a}$, where m_{var} is the variable mass $m_{\text{var}} = \gamma m$, except in the special case that $\mathbf{F}$ happens to be perpendicular to $\mathbf{v}$. In general, $\mathbf{F}$ and $\mathbf{a}$ are not even in the same direction.

First, $\mathbf{F} = d\mathbf{p}/dt = d(\gamma m\mathbf{v})/dt = \gamma m\mathbf{a} + (d\gamma/dt)m\mathbf{v}$. To evaluate $d\gamma/dt$, note that $\gamma = E/mc^2$ and, by the work-KE theorem (15.101), $dE/dt = \mathbf{F} \cdot \mathbf{v}$. Combining all of these, we get $\mathbf{F} = \gamma m\mathbf{a} + (\mathbf{F} \cdot \mathbf{v})\mathbf{v}/c^2$.

15.80 ⋆ A particle of mass m and charge q moves in a uniform, constant magnetic field $\mathbf{B}$. Show that if $\mathbf{v}$ is perpendicular to $\mathbf{B}$, the particle moves in a circle of radius

$$r = |\mathbf{p}/qB|. \tag{15.154}$$

[This result agrees with the nonrelativistic result (2.81), except that $\mathbf{p}$ is now the relativistic momentum $\mathbf{p} = \gamma m\mathbf{v}$.]

15.81 ⋆ An electron (mass 0.5 MeV/c^2) moves with speed $0.7c$ in a circular path in a magnetic field of 0.02 teslas. Using the relativistic result (15.154) of Problem 15.80, find the radius of the electron's orbit. What would your answer have been if you used the classical definition of momentum? [Needless to say, the relativistic result is confirmed by experiment, and this gave some of the first evidence of the correctness of relativistic mechanics.[1]]

The electron mass is $m \approx 0.5\ \text{MeV}/c^2 \approx 9 \times 10^{-31}$ kg, and with $v = 0.7c$, $\gamma = 1.4$. Thus

$$r = \left|\frac{\gamma m v}{qB}\right| = \frac{1.4 \times (9 \times 10^{-31}\ \text{kg}) \times (0.7 \times 3 \times 10^{8}\ \text{m/s})}{(1.6 \times 10^{-19}\ \text{C}) \times (0.02\ \text{T})} = 8.3\ \text{cm}.$$

The nonrelativistic answer is less by a factor of γ, $r(\text{nonrel}) = 5.9$ cm.

15.83 ⋆ Starting from the definition (15.100) of the force $\mathbf{F}$ on an object, prove that the transformation of the components of $\mathbf{F}$ as we pass from a frame $\mathcal{S}$ to a second frame $\mathcal{S}'$, traveling at speed V in the standard configuration relative to $\mathcal{S}$, is

$$F_1' = \frac{F_1 - \beta\mathbf{F} \cdot \mathbf{v}/c}{1 - \beta v_1/c}, \qquad F_2' = \frac{F_2}{\gamma(1 - \beta v_1/c)}, \qquad F_3' = \frac{F_3}{\gamma(1 - \beta v_1/c)} \tag{15.155}$$

where $\beta = \beta(V)$ and $\gamma = \gamma(V)$ relate to the relative speed of the two frames and $\mathbf{v}$ is the velocity of the object as measured in $\mathcal{S}$.

The force in frame $\mathcal{S}$ is $\mathbf{F} = d\mathbf{p}/dt$ and that in $\mathcal{S}'$ is $\mathbf{F}' = d\mathbf{p}'/dt'$. To relate these we have only to use the Lorentz transformation:

$$dp_1' = \gamma(dp_1 - \beta dE/c), \quad dp_2' = dp_2, \quad dp_3' = dp_3, \quad \text{and} \quad dt' = \gamma(dt - \beta dx_1/c).$$

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[1] In a paper in *Göttingen Nachrichten*, p. 143 (1901), Walter Kaufmann showed that the electron's "apparent mass" (what we would call its variable mass) in a magnetic field seemed to increase with speed in rough accord with the relativistic formula $m_{\text{var}} = \gamma(v)m$. Notice that this predated Einstein's first paper on relativity by some four years.

Thus

$$F_1' = \frac{dp_1'}{dt'} = \frac{\gamma(dp_1 - \beta dE/c)}{\gamma(dt - \beta dx_1/c)} = \frac{F_1 - \beta(dE/dt)/c}{1 - \beta v_1/c} = \frac{F_1 - \beta \mathbf{F} \cdot \mathbf{v}/c}{1 - \beta v_1/c}$$

where in the last equality I used the work-KE theorem to replace dE/dt by $\mathbf{F} \cdot \mathbf{v}$. Similarly

$$F_2' = \frac{dp_2'}{dt'} = \frac{dp_2}{\gamma(dt - \beta dx_1/c)} = \frac{F_2}{\gamma(1 - \beta v_1/c)} .$$

with a similar result for F_3'.

15.85 ⋆⋆ We have seen that there are processes in which the mass of an object varies with time. **(a)** Starting from (15.85), prove that $dm/dt_o = -u \cdot K/c^2$, where t_o is the object's proper time, u is its four-velocity, and K is the four-force on the object. **(b)** This means that the necessary and sufficient condition that a force doesn't change an object's mass is that $u \cdot K = 0$. It is an experimental fact that if a charged particle is at rest in an electromagnetic field (even instantaneously) then $dE/dt = 0$. Use this to argue that electromagnetic forces do not cause a particle's mass to change.

(a) Differentiating the relation $E^2 = (\mathbf{p}c)^2 + (mc^2)^2$ we find that

$$E\frac{dE}{dt_o} = \mathbf{p}c^2 \cdot \frac{d\mathbf{p}}{dt_o} + mc^4\frac{dm}{dt_o}$$

whence

$$-c^2\frac{dm}{dt_o} = \frac{\mathbf{p}}{m} \cdot \frac{d\mathbf{p}}{dt_o} - \frac{E}{mc^2}\frac{dE}{dt_o} = \gamma\mathbf{v} \cdot \frac{d\mathbf{p}}{dt_o} - \gamma\frac{dE}{dt_o} = \gamma(\mathbf{v}, c) \cdot \frac{d}{dt_o}(\mathbf{p}, E/c) = u \cdot K.$$

(b) Because $u \cdot K$ is invariant, we can evaluate it in any frame. In particular, in the particle's instantaneous rest frame $u = (\mathbf{0}, c)$ and $K = (d/dt)(\mathbf{p}, E/c)$, so $u \cdot K = dE/dt$, which we have been told is zero. Therefore the mass of a charged particle exposed only to electromagnetic fields is constant.

SECTION 15.16 Massless Particles; the Photon

15.87 ⋆ A neutral pion (Problem 15.86) is traveling with speed v when it decays into two photons, which are seen to emerge at equal angles θ on either side of the original velocity. Show that $v = c\cos\theta$.

Let's take take the direction of the initial pion to be our x axis and the plane of the two emerging photons to be the xy plane. Conservation of four-momentum implies that $p_\pi = p_{\gamma 1} + p_{\gamma 2}$, the y component of which says that $|\mathbf{p}_{\gamma 1}|\sin\theta = |\mathbf{p}_{\gamma 2}|\sin\theta$. Therefore $|\mathbf{p}_{\gamma 1}| = |\mathbf{p}_{\gamma 2}|$ and $E_{\gamma 1} = E_{\gamma 2} = |\mathbf{p}_{\gamma 1}|c$ (the last because the photon is massless). The x and t components of momentum conservation now tell us that

$$|p_\pi| = 2|\mathbf{p}_{\gamma 1}|\cos\theta \qquad \text{and} \qquad E_\pi = 2E_{\gamma 1} = 2|\mathbf{p}_{\gamma 1}|c,$$

Dividing the first of these by the second we find that $v_\pi = |p_\pi|c^2/E_\pi = c\cos\theta$.

15.89 ⋆ Show that any two zero-mass particles have a CM frame, provided their three-momenta are not parallel. [*Hint:* As you should explain, this is equivalent to showing that the sum of two forward light-like vectors is forward time-like, unless the spatial parts are parallel.]

Two particles have a CM frame if and only if there exists a frame where their total three-momentum is zero, and this is the case if and only if their total four-momentum is time-like. Now, a four-vector p is forward time-like if and only if $p_4 > |\mathbf{p}|$ (and forward light-like if and only if $p_4 = |\mathbf{p}| > 0$). Since our two particles are massless their four-momenta are forward light-like, so $p_{a4} = |\mathbf{p}_a| > 0$ and $p_{b4} = |\mathbf{p}_b| > 0$. Therefore their total momentum satisfies

$$p_4 = p_{a4} + p_{b4} = |\mathbf{p}_a| + |\mathbf{p}_b| \geq |\mathbf{p}_a + \mathbf{p}_b| = |\mathbf{p}|.$$

This proves that $p = p_a + p_b$ is forward time-like, *provided* the "$\geq$" sign is actually "$>$". Now, we know for any two three-vectors **A** and **B**, that $|\mathbf{A}| + |\mathbf{B}| > |\mathbf{A} + \mathbf{B}|$ unless **A** and **B** are parallel. Therefore p is time-like unless $\mathbf{p}_a$ and $\mathbf{p}_b$ are parallel.

15.91 ⋆⋆ An excited state X* of an atom at rest drops to its ground state X by emitting a photon. In atomic physics it is usual to assume that the energy E_γ of the photon is equal to the difference in energies of the two atomic states, $\Delta E = (M^* - M)c^2$, where M and M^* are the rest masses of the ground and excited states of the atom. This cannot be exactly true, since the recoiling atom X must carry away some of the energy ΔE. Show that in fact $E_\gamma = \Delta E[1 - \Delta E/(2M^*c^2)]$. Given that ΔE is of order a few eV, while the lightest atom has M of order 1 GeV/c^2, discuss the validity of the assumption that $E_\gamma = \Delta E$.

In the process $X^* \to X + \gamma$, the initial X* is at rest so has four-momentum $p^* = (\mathbf{0}, M^*c)$. The four-momentum of the outgoing photon is $p_\gamma = (\mathbf{p}_\gamma, E_\gamma/c)$. By conservation of four-momentum $p^* = p + p_\gamma$ or $p = p^* - p_\gamma$, which, if we square both sides, implies that

$$-M^2c^2 = -M^{*2}c^2 - 2p^* \cdot p_\gamma + 0 = -M^{*2}c^2 + 2M^*E_\gamma$$

or, solving for E_γ,

$$E_\gamma = \frac{M^{*2} - M^2}{2M^*}c^2 = (M^* - M)c^2\frac{(M^* + M)}{2M^*} = \Delta E\left(1 - \frac{\Delta E}{2M^*c^2}\right)$$

where in the last equality I used the fact that $M = M^* - \Delta E/c^2$. The fractional difference between ΔE and E_γ is $\Delta E/(2M^*c^2) \approx$ (a few eV)/(at least 2 GeV), of order one part in a billion or less.

15.93 ⋆⋆⋆ Consider a head-on elastic collision between a high-energy electron (energy E_o and speed $\beta_\text{o}c$) and a photon of energy $E_{\gamma\text{o}}$. Show that the final energy E_γ of the photon is

$$E_\gamma = E_\text{o}\frac{1 + \beta_\text{o}}{2 + (1 - \beta_\text{o})E_\text{o}/E_{\gamma\text{o}}}.$$

[*Hint:* Use (15.123).] Show that $E_\gamma < E_\text{o}$, but that if $\beta_\text{o} \to 1$, then $E_\gamma/E_\text{o} \to 1$; that is, a very high-energy electron loses almost all its energy to the photon in a head-on collision. What fraction of its original energy would the electron retain if $E_\text{o} \approx 10$ TeV and the photon was in the visible range, $E_{\gamma\text{o}} \approx 3$ eV? (Remember that the mass of the electron is about 0.5 MeV/c^2; 1 TeV = 10^{12} eV.)

The four-momenta of the initial and final electron and photon can be written as follows:

	initial	final
electron	$p_o = (\beta_o, 0, 0, 1)E_o/c$	p
photon	$p_{\gamma o} = (-1, 0, 0, 1)E_{\gamma o}/c$	$p_\gamma = (1, 0, 0, 1)E_\gamma/c$

According to (15.123), $p_o \cdot (p_{\gamma o} - p_\gamma) = p_{\gamma o} \cdot p_\gamma$, which becomes

$$E_o E_{\gamma o}(-\beta_o - 1) - E_o E_\gamma(\beta_o - 1) = E_{\gamma o} E_\gamma(-1 - 1).$$

Solving for E_γ we get the advertised answer

$$E_\gamma = E_o \frac{1 + \beta_o}{2 + (1 - \beta_o)E_o/E_{\gamma o}}. \tag{S15.7}$$

The numerator of the fraction on the right of Eq. (S15.7) is clearly less than 2, while the denominator is greater than 2. Therefore $E_\gamma < E_o$; the energy of the outgoing photon is less than the energy of the incident electron. But as $E_o \to \infty$ the proportion given to the photon approaches 100%. To see this, note that as $E_o \to \infty$, $\beta_o \to 1$ and the numerator in (S15.7) approaches 2. The denominator is a bit harder, because the factor $(1 - \beta_o)$ approaches zero while the factor E_o approaches infinity. However,

$$(1 - \beta_o)E_o = (1 - \beta_o)\frac{mc^2}{\sqrt{1 - \beta_o^2}} = mc^2\sqrt{\frac{1 - \beta_o}{1 + \beta_o}} \to 0. \tag{S15.8}$$

Thus the denominator approaches 2 and the ratio $E_\gamma/E_o \to 1$. In the limit $E_o \to \infty$ the proportion of the electron's energy transferred to the photon approaches 100%.

Before we put in numbers, let's rewrite Eq. (S15.7) as $E_\gamma = E_o(1 + \beta_o)/(2 + \xi)$, where according to Eq. (S15.8)

$$\xi = \frac{mc^2}{E_{\gamma o}}\sqrt{\frac{1 - \beta_o}{1 + \beta_o}}.$$

With the given numbers, $\beta_o = 1 - 1.25 \times 10^{-15}$ and $\xi = 0.004167$. Thus

$$E = E_o + E_{\gamma o} - E_\gamma = E_o \frac{1 - \beta_o + \xi}{2 + \xi} + E_{\gamma o}.$$

Because both $(1 - \beta_o)$ and $E_{\gamma o}/E_o$ are much much less than ξ, this gives (almost exactly) $E/E_o = \xi/(2 + \xi) = 0.002 = 0.2\%$.

SECTION 15.17 Tensors

15.95 ⋆ By making suitable choices for the n-dimensional vectors **a** and **b**, show that if $\tilde{\mathbf{a}}\mathbf{C}\mathbf{b} = \tilde{\mathbf{a}}\mathbf{D}\mathbf{b}$ for any choices of **a** and **b** (where **C** and **D** are $n \times n$ matrices), then $\mathbf{C} = \mathbf{D}$.

The condition $\tilde{\mathbf{a}}\mathbf{C}\mathbf{b} = \tilde{\mathbf{a}}\mathbf{D}\mathbf{b}$ is equivalent to the condition $\tilde{\mathbf{a}}(\mathbf{C} - \mathbf{D})\mathbf{b} = 0$. Thus what we have to prove is that if $\tilde{\mathbf{a}}\mathbf{E}\mathbf{b} = 0$ for all vectors **a** and **b**, then $\mathbf{E} = 0$. To prove this, choose **a** and **b** with the form below with the single nonzero entry of **a** in the kth row and that of **b** in the lth row. Then it is easy to see that $\tilde{\mathbf{a}}\mathbf{E}\mathbf{b} = E_{kl}$. Since this has to be zero for any choice of k and l, we conclude that $\mathbf{E} = 0$.

$$\mathbf{a} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

15.97 ⋆ **(a)** A tensor T is said to be symmetric if $T_{\mu\nu} = T_{\nu\mu}$. Prove that if T is symmetric in one inertial frame, then it is symmetric in all inertial frames. **(b)** T is antisymmetric if $T_{\mu\nu} = -T_{\nu\mu}$. Prove that if T is antisymmetric in one inertial frame, then it is antisymmetric in all inertial frames. (An example of the latter property is the electromagnetic field tensor, which is antisymmetric in all frames.)

(a) Suppose that T is symmetric, that is, $\tilde{T} = T$. Then $(T')\tilde{} = (\Lambda T \tilde{\Lambda})\tilde{} = \Lambda T \tilde{\Lambda} = T'$, where I have used the rules that $(AB)\tilde{} = \tilde{B}\tilde{A}$ and that $(\tilde{A})\tilde{} = A$. This shows that T' is also symmetric.
(b) By the same argument, if $\tilde{T} = -T$, then also $(T')\tilde{} = -T'$.

15.99 ⋆⋆ A useful form of the quotient rule for three-dimensional vectors is this: Suppose that **a** and **b** are known to be three-vectors and suppose that for every orthogonal set of axes there is a 3×3 matrix **T** with the property that $\mathbf{b} = \mathbf{Ta}$ for every choice of **a**, then **T** is a tensor. **(a)** Prove this. **(b)** State and prove the corresponding rule for four-vectors and four-tensors.

(a) That **a** and **b** are vectors means that, given any two frames $\mathcal{S}$ and $\mathcal{S}'$, related by a rotation **R**, $\mathbf{a}' = \mathbf{Ra}$ and $\mathbf{b}' = \mathbf{Rb}$ (or, in reverse, $\mathbf{a} = \tilde{\mathbf{R}}\mathbf{a}'$, etc.). Thus $\mathbf{b}' = \mathbf{T}'\mathbf{a}'$, but also $\mathbf{b}' = \mathbf{Rb} = \mathbf{RTa} = \mathbf{RT}\tilde{\mathbf{R}}\mathbf{a}'$. Since these two expressions for $\mathbf{b}'$ must be equal for any choice of $\mathbf{a}'$, it follows that $\mathbf{T}' = \mathbf{RT}\tilde{\mathbf{R}}$; that is, **T** is a tensor.
(b) Suppose that p and q are known to be four-vectors and suppose that for every inertial frame there is a 4×4 matrix T with the property that $q = T \cdot p$ for every choice of p, then T is a four-tensor. The proof parallels the three-dimensional case, except that where $p' = \Lambda p$, $p = G\tilde{\Lambda}Gp'$. Then $q' = T' \cdot p' = T'Gp'$, but also $q' = \Lambda q = \Lambda T \cdot p = \Lambda T G p = \Lambda T G(G\tilde{\Lambda}Gp') = \Lambda T \tilde{\Lambda} G p'$. Since these are equal for any choice of p' we see that $T' = \Lambda T \tilde{\Lambda}$; that is, T is a tensor.

SECTION 15.18 Electrodynamics and Relativity

15.101 ⋆ **(a)** Prove that $\mathbf{E} \cdot \mathbf{B}$ and $E^2 - c^2B^2$ are both invariant under any Lorentz tranformation. [Use the transformation equations (15.146) to prove the required results for the standard boost and then explain why if either quantity is invariant under the standard boost then it is invariant under *any* Lorentz transformation.] Use these results to prove the following two propositions: **(b)** If **E** and **B** are perpendicular in frame $\mathcal{S}$, then they are perpendicular in any other frame $\mathcal{S}'$, and **(c)** if $E > cB$ in a frame $\mathcal{S}$, then there cannot exist a frame in which $E = 0$.

(a) If $\mathcal{S}$ is obtained from $\mathcal{S}'$ by the standard boost, then, according to (15.146)

$$\begin{aligned} \mathbf{E}' \cdot \mathbf{B}' &= E_1'B_1' + E_2'B_2' + E_3'B_3' \\ &= E_1B_1 + \gamma^2(E_2 - \beta c B_3)(B_2 + \beta E_3/c) + \gamma^2(E_3 + \beta c B_2)(B_3 - \beta E_2/c) \\ &= E_1B_1 + E_2B_2 + E_3B_3 = \mathbf{E} \cdot \mathbf{B} \end{aligned}$$

[Remember that $\gamma^2(1 - \beta^2) = 1$.] Similarly

$$\begin{aligned} E'^2 &= E_1'^2 + E_2'^2 + E_3'^2 = E_1^2 + \gamma^2(E_2 - \beta c B_3)^2 + \gamma^2(E_3 + \beta c B_2)^2 \\ &= E_1^2 + \gamma^2(E_2^2 + E_3^2) + \gamma^2\beta^2c^2(B_2^2 + B_3^2) - 2\gamma^2\beta c(E_2B_3 - E_3B_2) \end{aligned}$$

and, as you can check,

$$c^2B'^2 = \gamma^2\beta^2(E_2^2 + e_3^2) + c^2\Big[B_1^2 + \gamma^2(B_2^2 + B_3^2)\Big] - 2\gamma^2\beta c(E_2B_3 - E_3B_2).$$

Taking the difference of these two and remembering that $\gamma^2(1 - \beta^2) = 1$, we find that

$$E'^2 - c^2B'^2 = E^2 - c^2B^2.$$

We have proved that neither of the named quantities is changed by a standard boost. They are clearly unchanged by any rotation, so, since any Lorentz transformation can be made up from rotations and a standard boost, they are actually invariant under any Lorentz transformation.
(b) If **E** and **B** are perpendicular in $\mathcal{S}$, then $\mathbf{E} \cdot \mathbf{B} = 0$, and by the invariance of $\mathbf{E} \cdot \mathbf{B}$ the same is true in any other frame.
(c) If $E > cB$ in $\mathcal{S}$, then $E^2 - c^2B^2 > 0$, and by the invariance of $E^2 - c^2B^2$ the same is true in any other frame.

15.103 ⋆ **(a)** Using the transformation equations (15.146), show that if $\mathbf{E} = 0$ in frame $\mathcal{S}$, then $\mathbf{E}' = \mathbf{v} \times \mathbf{B}'$ in $\mathcal{S}'$. **(b)** Similarly, show that if $\mathbf{B} = 0$ in frame $\mathcal{S}$, then $\mathbf{B}' = -\mathbf{v} \times \mathbf{E}'/c^2$.

(a) If $\mathbf{E} = 0$, then, according to (15.146)

$$B_1' = B_1, \quad B_2' = \gamma B_2, \quad B_3' = \gamma B_3$$

and

$$E_1' = 0, \quad E_2' = -\gamma\beta c B_3 = -vB_3', \quad E_3' = \gamma\beta c B_2 = -vB_2'.$$

Combining these last three (and recalling that **v** is along the x axis), we see that $\mathbf{E}' = \mathbf{v} \times \mathbf{B}'$.
(b) In exactly the same way, if $\mathbf{B} = 0$, then $\mathbf{B}' = -\mathbf{v} \times \mathbf{E}'/c^2$.

15.105 ⋆⋆ Since $\mathcal{F}$ is a four-tensor it has to transform according to the rule (15.145), $\mathcal{F}' = \Lambda\mathcal{F}\tilde{\Lambda}$. Using the form (15.143) for $\mathcal{F}$ and the standard Lorentz boost for Λ, find the matrix $\mathcal{F}'$ and verify the transformation equations (15.146) for the electromagnetic fields.

The matrices Λ and $\mathcal{F}$ are given in Eqs. (15.43) and (15.143). Using them you can easily check that

$$\Lambda\mathcal{F}\tilde{\Lambda} = \begin{bmatrix} 0 & \gamma(B_3 - \beta E_2/c) & -\gamma(B_2 + \beta E_3/c) & -E_1/c \\ \cdot & 0 & B_1 & -\gamma(E_2 - \beta c B_3)/c \\ \cdot & \cdot & 0 & -\gamma(E_3 + \beta c B_2)/c \\ \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

(I didn't bother to write the elements below the diagonal because we know that the matrix is antisymmetric.) But this must be equal to $\mathcal{F}'$,

$$\mathcal{F}' = \begin{bmatrix} 0 & B_3' & -B_2' & -E_1'/c \\ \cdot & 0 & B_1' & -E_2'/c \\ \cdot & \cdot & 0 & -E_3'/c \\ \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

Comparing these two you can read off the transformed fields as in Eqs. (15.146).

15.107 ⋆⋆ It is a result well known in classical electromagnetism that one can introduce a three-scalar potential ϕ and a three-vector potential $\mathbf{A}$ such that the fields $\mathbf{E}$ and $\mathbf{B}$ can be written as

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \qquad \text{and} \qquad \mathbf{B} = \nabla \times \mathbf{A}. \tag{15.157}$$

In relativity these potentials are combined to form a single four-potential $A = (\mathbf{A}, \phi/c)$. Prove that

$$\mathcal{F}_{\mu\nu} = \Box_\mu A_\nu - \Box_\nu A_\mu$$

where $\Box$ is the four-dimensional gradient operator defined in (15.156) of Problem 15.100. (If we accept that A really is a four-vector, this gives an alternative proof that $\mathcal{F}$ is a four tensor.)

We are given that $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$, $\mathbf{B} = \nabla \times \mathbf{A}$ and $A = (\mathbf{A}, \phi/c)$. I'll examine the components of $\mathcal{F}$ in turn. First, if $i = 1, 2$, or 3,

$$\mathcal{F}_{i4} = -\frac{E_i}{c} = \frac{\partial}{\partial x_i}\frac{\phi}{c} + \frac{1}{c}\frac{\partial A_i}{\partial t} = \frac{\partial}{\partial x_i}A_4 + \frac{\partial A_i}{\partial x_4} = \Box_i A_4 - \Box_4 A_i.$$

and

$$\mathcal{F}_{12} = B_3 = \frac{\partial}{\partial x_1}A_2 - \frac{\partial}{\partial x_2}A_1 = \Box_1 A_2 - \Box_2 A_1.$$

All other elements follow similarly, and we've proved that $\mathcal{F}_{\mu\nu} = \Box_\mu A_\nu - \Box_\nu A_\mu$, for any μ and ν equal to 1,2,3, or 4.

15.109 ⋆⋆⋆ Two equal charges q are moving side-by-side in the positive x direction in frame $\mathcal{S}$. The distance between them is r and their speed is v. Find the force on either charge due to the other in two ways: **(a)** Find the force in their rest frame $\mathcal{S}'$ and transform back to $\mathcal{S}$, using the force transformation (15.155) of Problem 15.83. Note that the force in $\mathcal{S}$ is less than in the rest frame. **(b)** Find the electric and magnetic fields in $\mathcal{S}'$ and thence in $\mathcal{S}$, using the field transformation (15.146). Use these fields (in $\mathcal{S}$) to write down the Lorentz force on either charge in $\mathcal{S}$. Note that in $\mathcal{S}$ there is an attractive electric force and a repulsive magnetic force. As $\beta \to 1$ they become nearly equal and their resultant approaches zero.

(a) In $\mathcal{S}'$ the two charges are at rest on the y axis a distance r apart (this transverse distance is the same in either frame), so $\mathbf{F}' = (0, kq^2/r^2, 0)$. The inverse of the force transformation (15.155) gives the force $\mathbf{F}$ measured in $\mathcal{S}$ as

$$F_1 = \frac{F_1' + \beta\mathbf{F}' \cdot \mathbf{v}'/c}{1 + \beta v_1'/c} = 0, \quad F_2 = \frac{F_2'}{\gamma(1 + \beta v_1'/c)} = \frac{F_2'}{\gamma} = \frac{kq^2}{\gamma r^2}, \quad F_3 = 0.$$

(b) The fields of the lower charge at the location of the upper one, as measured in $\mathcal{S}'$, are $\mathbf{E}' = (0, kq/r^2, 0)$ and $\mathbf{B}' = 0$. Using the inverse field transformations, we find the fields in $\mathcal{S}$ to be $\mathbf{E} = (0, \gamma kq/r^2, 0)$ and $\mathbf{B} = (0, 0, \gamma\beta kq/cr^2)$. (Note well how each charge feels a magnetic field due to the other moving charge.) Given that the velocity of either particle in $\mathcal{S}$ is $\mathbf{v} = (v, 0, 0)$, we can now evaluate the foce on the upper charge as

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \frac{\gamma kq^2}{r^2}\hat{\mathbf{y}} - \frac{v\gamma\beta kq^2}{cr^2}\hat{\mathbf{y}} = \frac{\gamma(1-\beta^2)kq^2}{r^2}\hat{\mathbf{y}} = \frac{kq^2}{\gamma r^2}\hat{\mathbf{y}}$$

since $\gamma(1 - \beta^2) = 1/\gamma$. This is the same as we found in part (a).

15.111 ⋆⋆⋆ Two of Maxwell's four equations read

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_o \mathbf{J} \qquad \text{and} \qquad \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_o}\varrho \tag{15.159}$$

where $\mathbf{J}$ and ϱ are the current and charge densities that gave rise to the fields. Show that these two equations can be written as the single four-vector equation $\Box \cdot \mathcal{F} = -\mu_o \tilde{J}$, where $\Box$ is the four-dimensional gradient operator introduced in Problem 15.100, J is the four-current $(\mathbf{J}, c\varrho)$, and the scalar product $\Box \cdot \mathcal{F} = \tilde{\Box} G \mathcal{F}$.

The proposed equation $\Box \cdot \mathcal{F} = \mu_o \tilde{J}$ is

$$\begin{aligned}
\Box \cdot \mathcal{F} = \tilde{\Box} G \mathcal{F} &= [\partial_x, \partial_y, \partial_z, \partial_t/c] \begin{bmatrix} 0 & B_3 & -B_2 & -E_1/c \\ -B_3 & 0 & B_1 & -E_2/c \\ B_2 & -B_1 & 0 & -E_3/c \\ E_1/c & E_2/c & E_3/c & 0 \end{bmatrix} \\
&= [-\partial_y B_z + \partial_z B_y + \partial_t E_x/c^2, \partial_x B_z - \partial_z B_x + \partial_t E_y/c^2, \\
&\quad - \partial_x B_y + \partial_y B_x + \partial_t E_z/c^2, -(\partial_x E_x + \partial_y E_y + \partial_z E_z)/c^2] \\
&= -[\mu_o J_x, \mu_o J_y, \mu_o J_z, \varrho/c\epsilon_o].
\end{aligned}$$

Here I have used the abbreviations $\partial_x = \partial/\partial x$ and so on. In writing the second equality, I noted that $\tilde{\Box} G = [\partial_x, \partial_y, \partial_z, \partial_t/c]$ and in the last line I used the fact that $\mu_o c = 1/c\epsilon_o$. If you look carefully at the last equality, you will recognize its first three components as the first of the Maxwell equations (15.159) and the time component as the second.

CHAPTER 16

Continuum Mechanics

PROBLEMS AND SOLUTIONS

SECTION 16.1 Motion of a Taut String

16.1 ⋆ Verify that the quantity $c = \sqrt{T/\mu}$ that appears in the wave equation for a string does indeed have the units of a speed.

The SI units of the tension T and linear density μ are $[T]$ = [force] = kg·m/s^2 and $[\mu]$ = [mass/length] = kg/m. Therefore, those of $c = \sqrt{T/\mu}$ are

$$[c] = \sqrt{\frac{\text{kg}\cdot\text{m}}{\text{s}^2}\cdot\frac{\text{m}}{\text{kg}}} = \frac{\text{m}}{\text{s}}$$

as required.

SECTION 16.2 The Wave Equation

16.3 ⋆ Let $f(\xi)$ be an arbitrary (twice differentiable) function. Show by direct substitution that $f(x - ct)$ is a solution of the wave equation (16.4).

Let $f'(\xi) = df(\xi)/d\xi$ and likewise $f''(\xi) = d^2 f(\xi)/d\xi^2$. By the chain rule

$$\frac{\partial}{\partial t} f(x - ct) = -cf'(x - ct) \qquad \text{and} \qquad \frac{\partial^2}{\partial t^2} f(x - ct) = c^2 f''(x - ct)$$

and

$$\frac{\partial}{\partial x} f(x - ct) = f'(x - ct) \qquad \text{and} \qquad \frac{\partial^2}{\partial x^2} f(x - ct) = f''(x - ct).$$

Comparing these two, we see that

$$\frac{\partial^2}{\partial t^2} f(x - ct) = c^2 \frac{\partial^2}{\partial x^2} f(x - ct)$$

which is the wave equation.

16.5 ⋆ **(a)** Show that $u = g(x + ct)$ is a solution of the wave equation (16.4) for any twice differentiable function $g(\xi)$. **(b)** Argue clearly that this solution represents a disturbance that travels undistorted to the left.

(a) Let $g'(\xi) = dg(\xi)/d\xi$ and likewise $g''(\xi) = d^2g(\xi)/d\xi^2$. By the chain rule

$$\frac{\partial}{\partial t}g(x + ct) = cg'(x + ct) \qquad \text{and} \qquad \frac{\partial^2}{\partial t^2}g(x + ct) = c^2 g''(x + ct)$$

and

$$\frac{\partial}{\partial x}g(x + ct) = g'(x + ct) \qquad \text{and} \qquad \frac{\partial^2}{\partial x^2}g(x + ct) = g''(x + ct).$$

Comparing these two, we see that

$$\frac{\partial^2}{\partial t^2}g(x + ct) = c^2 \frac{\partial^2}{\partial x^2}g(x + ct)$$

which is the wave equation.

(b) If $u(x, t) = g(x + ct)$, then at time zero, $u(x, 0) = g(x)$. Now suppose, for example, that the function $g(x)$ has a crest (or trough, or any other feature) at the point $x = x_o$. Then at a later time t, $u(x, t) = g(x + ct)$ has the same crest at the point $x + ct = x_o$ or $x = x_o - ct$. That is, as time goes by, the crest moves to the left with speed c. Because the same applies to all other features, we conclude that the whole initial disturbance moves bodily to the left with speed c.

16.7 ⋆⋆ [Computer] Make plots of the two triangular waves of Example 16.1 (page 686) at several closely spaced times and then animate them. Describe the motion. For the purposes of the plot you may as well take the speed c, the height of the triangle at time 0, and the half width of the base all equal to 1. Make your plots for lots of times ranging from $t = -4$ to 4.

In Figure S16.7 is a subset of the requested graphs. Initially ($t < -1$), the two separate triangles move inward from the left and right. At $t = -1$ they start to overlap and reinforce one another, producing a truncated triangle. At $t = 0$ they overlap completely, producing a triangle of twice their separate heights, and then they start to move apart again.

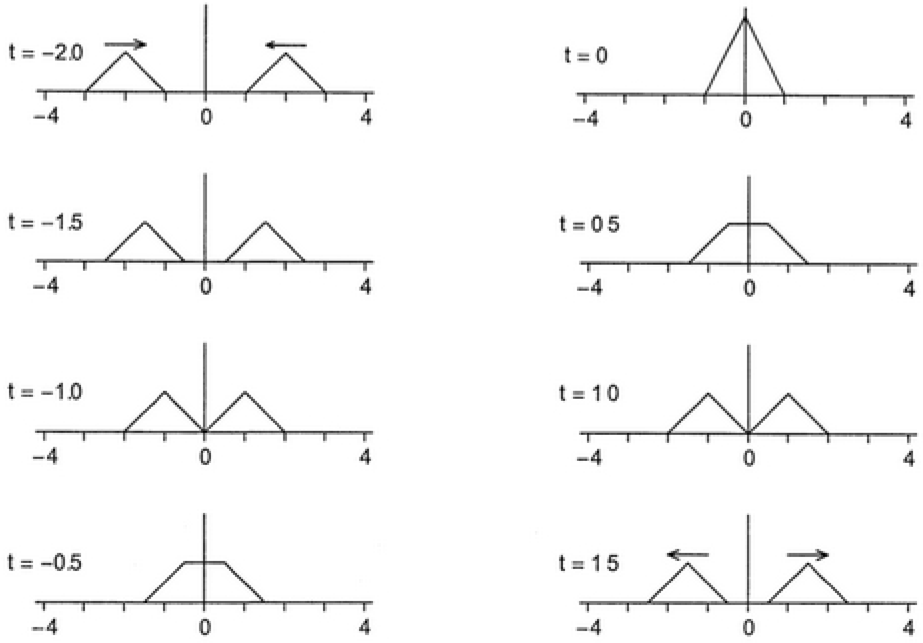

Figure S16.7

SECTION 16.3 Boundary Conditions; Waves on a Finite String

16.9 ⋆⋆ The motion of a finite string, fixed at both ends, was determined by the wave equation (16.19) and the boundary conditions (16.20). We solved these by looking for a solution that was sinusoidal in time. A different, and rather more general, approach to problems of this kind is called **separation of variables**. In this approach, we seek solutions of (16.19) with the *separated* form $u(x,t) = X(x)T(t)$, that is, solutions that are a simple product of one function of x and a second of t. [As usual, there's nothing to stop us trying to find a solution of this form. In fact, there is a large class of problems (including this one) where this approach is known to produce solutions, and enough solutions to allow expansion of *any* solution.] **(a)** Substitute this form into (16.19) and show that you can rewrite the equation in the form $T''(t)/T(t) = c^2X''(x)/X(x)$. **(b)** Argue that this last equation requires that both sides of this equation are separately equal to the same constant (call it K). It can be shown that K has to be negative.[1] Use this to show that the function $T(t)$ has to be sinusoidal — which establishes (16.21) and we're back to the solution of Section 16.3. The method of separation of variables plays an important role in several areas, notably quantum mechanics and electromagnetism.

(a) We are certainly free to try for a solution of the form $u(x,t) = X(x)T(t)$. If we substitute this into Eq. (16.19), $\partial^2u/\partial t^2 = c^2\partial^2u/\partial x^2$, we find $X(x)T''(t) = c^2T(t)X''(x)$, from which it follows that

$$\frac{T''(t)}{T(t)} = c^2\frac{X''(x)}{X(x)} \tag{S16.1}$$

where, as usual, primes indicate differentiation with respect to the argument. This equation must hold for all t and all x in the ranges of interest. If we temporarily fix x, then Eq. (S16.1) says that the left side is independent of t. Similarly, if we fix t, we see that the right side is independent of x. Since the two sides are equal, this implies that both sides are equal to a constant K, independent of t and x. As discussed in the footnote, this constant has to be negative, so we can call it $-\omega^2$. Therefore, $T''(t) = -\omega^2T(t)$, which implies that $T(t) = A\cos(\omega t - \delta)$ and hence that $u(x,t)$ has the form (16.21).

16.11 ⋆⋆ [Computer] Make plots similar to Figure 16.8 of the wave of Example 16.2 but from $t = 0$ to τ, the period, and for more closely spaced times. Animate your pictures and describe the motion.

Here is the story from $t = \tau/2$ to τ, at the same intervals as in Fig. 16.8. At $t = \tau/2$ the two traveling waves have formed a single inverted triangle. They then separate and move out toward the walls, where they reflect, invert, and travel inward again. By the time $t = \tau$ they have coalesced and the string is back to its original position.

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[1] Actually this isn't hard to show. Look at the equation $X''(x)/X(x) = K/c^2$. You can show that if $K > 0$ there are no solutions satisfying the boundary conditions that $X(0) = X(L) = 0$.

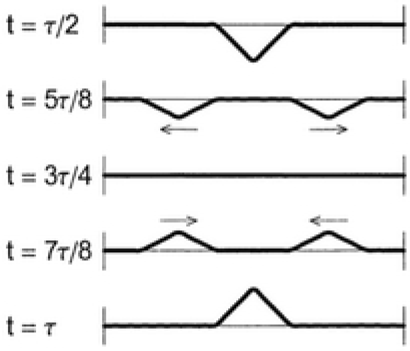

Figure S16.11

16.13 ⋆⋆ In connection with Equation (16.31), I claimed that *any* function on the interval $0 \le x \le L$ can be expanded in a Fourier series containing just sine functions. This is at first sight very surprising since one is used to the claim that the general Fourier series requires sines *and* cosines. In this problem, you'll prove this surprising claim. Let $f(x)$ be any function defined for $0 \le x \le L$. We can define a function $f(x)$ for *all* x by setting it equal to the given function in the original interval and requiring that

$$f(-x) = -f(x) \qquad \text{and} \qquad f(x+2L) = f(x). \tag{16.143}$$

for all x. Prove that this defines a function which is (1) periodic with period $2L$, (2) odd, and (3) the same as the original $f(x)$ on the original interval. Write down the ordinary Fourier expansion for this new $f(x)$ and show that the coefficients of the cosine terms are all zero. This establishes the possibility of expanding the original function in terms of sines alone.[2] Bearing in mind that the period of the new function is $2L$, write down the standard formula (5.84) for the expansion coefficients and show that your answer agrees with (16.33). The Fourier sine series is especially convenient for discussing functions that are zero at the end points $x = 0$ and L.

The first condition in (16.143) defines $f(x)$ in the interval $-L \le x \le 0$, and the second condition then defines it for all x. The second condition guarantees that it is $2L$-periodic and the first condition that it is odd. Clearly the new $f(x)$ coincides with the old in the original interval $0 \le x \le L$. Since the new function is periodic with period $\lambda = 2L$, it can be expanded in a standard Fourier series as

$$f(x) = \sum_{n=0}^{\infty} (A_n \cos k_n x + B_n \sin k_n x).$$

However, since we cunningly defined $f(x)$ to be an odd function, the coefficients A_n of the cosine terms are all zero, and since the period is $\lambda = 2L$ the wave numbers k_n are $2n\pi/\lambda = n\pi/L$ as in (16.27). This establishes the claimed expansion (16.31). Finally the usual integral for the coefficients is $B_n = (2/\lambda)\int_{-L}^{L} f \sin(k_n x)dx$. If we make the replacement $\lambda = 2L$, the factors of 2 cancel, but we can then replace the integral from $-L$ to L by twice the integral from 0 to L, and we wind up with $B_n = (2/L)\int_0^L f \sin(k_n x)dx$ as in (16.33).

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[2] But note that it has the form $\sum B_n \sin(n\pi x/L)$. The usual Fourier series has sines and cosines, but their argument is $2n\pi x/L$. Thus the new Fourier sine series has, in a sense, twice as many terms to make up for having only sines.

SECTION 16.4 The Three-Dimensional Wave Equation

16.15 ★★ Let $f(\xi)$ be any function with first two derivatives $f'(\xi)$ and $f''(\xi)$, and let $\mathbf{n}$ be an arbitrary fixed unit vector. **(a)** Show that $\nabla f(\mathbf{n}\cdot\mathbf{r}-ct)=\mathbf{n}f'(\mathbf{n}\cdot\mathbf{r}-ct)$. **(b)** Hence show that $f(\mathbf{n}\cdot\mathbf{r}-ct)$ satisfies the three-dimensional wave equation (16.39). **(c)** Argue that $f(\mathbf{n}\cdot\mathbf{r}-ct)$ represents a signal that is constant in any plane perpendicular to $\mathbf{n}$ (at any fixed time t) and propagates rigidly with speed c in the direction of $\mathbf{n}$.

(a) By the chain rule,

$$\frac{\partial}{\partial x}f(\mathbf{n}\cdot\mathbf{r}-ct)=f'(\mathbf{n}\cdot\mathbf{r}-ct)\frac{\partial}{\partial x}(\mathbf{n}\cdot\mathbf{r})=n_x f'(\mathbf{n}\cdot\mathbf{r}-ct). \tag{S16.2}$$

Combining this with the two corresponding results for $\partial f/\partial y$ and $\partial f/\partial z$, we conclude that $\nabla f(\mathbf{n}\cdot\mathbf{r}-ct)=\mathbf{n}f'(\mathbf{n}\cdot\mathbf{r}-ct)$.

(b) Differentiating Eq. (S16.2) again we see that

$$\frac{\partial^2}{\partial x^2}f(\mathbf{n}\cdot\mathbf{r}-ct)=n_x^2 f''(\mathbf{n}\cdot\mathbf{r}-ct)$$

and adding this to the two corresponding results for the y and z derivatives, we find that $\nabla^2 f(\mathbf{n}\cdot\mathbf{r}-ct)=f''(\mathbf{n}\cdot\mathbf{r}-ct)$, because $n_x^2+n_y^2+n_z^2=1$. On the other hand, differentiating twice with respect to t, we find

$$\frac{\partial^2}{\partial t^2}f(\mathbf{n}\cdot\mathbf{r}-ct)=c^2 f''(\mathbf{n}\cdot\mathbf{r}-ct)=c^2\nabla^2 f(\mathbf{n}\cdot\mathbf{r}-ct)$$

by the previous result. This is the wave equation.

(c) At any fixed time t, $f(\mathbf{n}\cdot\mathbf{r}-ct)$ is constant provided $\mathbf{n}\cdot\mathbf{r}=$ const; that is, $f(\mathbf{n}\cdot\mathbf{r}-ct)$ is constant in any plane perpendicular to $\mathbf{n}$. Suppose now that $f(\xi)$ has a maximum (or any other feature) at $\xi=\xi_o$. Then $f(\mathbf{n}\cdot\mathbf{r}-ct)$ will share this feature at any point satisfying $\mathbf{n}\cdot\mathbf{r}=\xi_o+ct$, and this defines a plane perpendicular to $\mathbf{n}$ that travels at speed c in the direction of $\mathbf{n}$.

SECTION 16.6 Stress and Strain: the Elastic Moduli

16.17 ★★ In Section 16.1 we derived the wave equation for transverse waves in a taut string. Here you will examine the possibility of longitudinal waves in the same string. Suppose that an element of string whose equilibrium position is x is displaced a short distance in the x direction to $x+u(x,t)$. **(a)** Consider a short piece of string of length l and use the definition (16.55) of Young's modulus YM to show that the tension is $F=A\,\mathrm{YM}\,\partial u/\partial x$, where A is the cross sectional area of the string. [If the string is already in tension in its equilibrium position, this F is the *additional* tension, that is, F = (actual − equilibrium).] **(b)** Now consider the forces on a short section of string dx and show that u obeys the wave equation with wave speed $c=\sqrt{\mathrm{YM}/\varrho}$ where ϱ is the density (mass/volume) of the string.

(a) The increase in length of a segment of length l near position x is $dl=u(x+l)-u(x)\approx l\,\partial u/\partial x$. Thus, according to (16.55) the (additional) tension in the string is $F=A\,\mathrm{YM}\,dl/l=A\,\mathrm{YM}\,\partial u/\partial x$.

(b) The net force on a segment dx is

$$F_{\text{net}}=F(x+dx)-F(x)=A\,\mathrm{YM}\left[\frac{\partial u}{\partial x}(x+dx,t)-\frac{\partial u}{\partial x}(x,t)\right]=A\,\mathrm{YM}\,\frac{\partial^2 u}{\partial x^2}dx$$

But by Newton's second law, $F_{\text{net}} = ma = (\varrho A dx)\partial^2 u/\partial t^2$, and comparing these two expressions, we find

$$\frac{\partial^2 u}{\partial t^2} = \frac{\text{YM}}{\varrho}\frac{\partial^2 u}{\partial x^2}$$

which is the wave equation with wave speed $c = \sqrt{\text{YM}/\varrho}$.

SECTION 16.7 The Stress Tensor

16.19 ⋆ Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be any two unit vectors and P a point in a continuous medium. $\mathbf{F}(\mathbf{n}_1\, dA)$ is the surface force on a small area dA at P with unit outward normal $\mathbf{n}_1$, so $\mathbf{n}_2 \cdot \mathbf{F}(\mathbf{n}_1\, dA)$ is the component of that force in the direction of $\mathbf{n}_2$. Prove *Cauchy's reciprocal theorem* that $\mathbf{n}_2 \cdot \mathbf{F}(\mathbf{n}_1\, dA) = \mathbf{n}_1 \cdot \mathbf{F}(\mathbf{n}_2\, dA)$.

The surface force on a small surface element $\mathbf{n}_1 dA$ is $\mathbf{F}(\mathbf{n}_1 dA) = \boldsymbol{\Sigma}\,\mathbf{n}_1 dA$ and the component of this force in the direction of $\mathbf{n}_2$ is $\mathbf{n}_2 \cdot (\boldsymbol{\Sigma}\,\mathbf{n}_1) dA$. To emphasize the matrix products concerned, I'll temporarily use an underline to indicate a matrix. (For example, the dot product $\mathbf{a} \cdot \mathbf{b}$ of two vectors becomes $\mathbf{a} \cdot \mathbf{b} = \underline{\tilde{\mathbf{a}}}\,\underline{\mathbf{b}}$, where $\underline{\mathbf{b}}$ is the 3×1 column representing the vector $\mathbf{b}$ and $\underline{\tilde{\mathbf{a}}}$ is the 1×3 row representing $\mathbf{a}$.) With this notation

$$\mathbf{n}_2 \cdot (\boldsymbol{\Sigma}\,\mathbf{n}_1) = \underline{\tilde{\mathbf{n}}}_2\,\underline{\boldsymbol{\Sigma}}\,\underline{\mathbf{n}}_1 = \underline{\tilde{\mathbf{n}}}_1\,\underline{\tilde{\boldsymbol{\Sigma}}}\,\underline{\mathbf{n}}_2 = \underline{\tilde{\mathbf{n}}}_1\,\underline{\boldsymbol{\Sigma}}\,\underline{\mathbf{n}}_2 = \mathbf{n}_1 \cdot (\boldsymbol{\Sigma}\,\mathbf{n}_2)$$

where for the second equality I used the fact that any number is equal to its transpose and that $(\underline{\mathbf{u}\mathbf{v}\mathbf{w}})^{\tilde{}} = \underline{\tilde{\mathbf{w}}\tilde{\mathbf{v}}\tilde{\mathbf{u}}}$, and for the third equality I use the symmetry of $\boldsymbol{\Sigma}$.

16.21 ⋆⋆ At any given point P of a continuous medium, the surface forces are given by the stress tensor, which is a real symmetric matrix $\boldsymbol{\Sigma}$. It is a well-known theorem of linear algebra (see the appendix) that any such matrix can be brought into diagonal form by a suitable rotation of the Cartesian coordinate axes. Use this to prove that at any point P there are three orthogonal directions (the **principal stress axes** at P) with the property that the surface force on any surface normal to one of these directions is exactly normal to the surface.

The theorem quoted guarantees that there is a set of Cartesian axes with respect to which $\boldsymbol{\Sigma}$ has the diagonal form shown below on the left. Now consider the surface force on a small element of area dA normal to the x axis of that set. The vector $d\mathbf{A}$ for this surface is $d\mathbf{A} = dA\,\hat{\mathbf{x}}$, whose matrix form is given in the middle expression below. Therefore the force on this small surface is as given on the right.

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}, \quad d\mathbf{A} = dA\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{F} = \boldsymbol{\Sigma}\, d\mathbf{A} = dA\begin{bmatrix} \sigma_{11} \\ 0 \\ 0 \end{bmatrix}.$$

We see that the force on a surface normal to $\hat{\mathbf{x}}$ is in the direction of $\hat{\mathbf{x}}$ (that is, normal to the surface). Since the same happens with surfaces normal to $\hat{\mathbf{y}}$ or $\hat{\mathbf{z}}$, we've proved the requested result.

SECTION 16.8 The Strain Tensor for a Solid

16.23 ⋆ An important tool in the development of the strain tensor was the decomposition (16.79) of a matrix $\mathbf{M}$ into its antisymmetric and symmetric parts. Prove that this decomposition is unique.

[*Hint:* Show that if $\mathbf{M} = \mathbf{M}_\mathrm{A} + \mathbf{M}_\mathrm{S}$ where $\mathbf{M}_\mathrm{A}$ and $\mathbf{M}_\mathrm{S}$ are respectively antisymmetric and symmetric, then $\mathbf{M}_\mathrm{A} = \frac{1}{2}(\mathbf{M} - \tilde{\mathbf{M}})$ and $\mathbf{M}_\mathrm{S} = \frac{1}{2}(\mathbf{M} + \tilde{\mathbf{M}})$.]

Suppose that $\mathbf{M} = \mathbf{M}_A + \mathbf{M}_S$, where $\mathbf{M}_A$ and $\mathbf{M}_S$ are respectively antisymmetric and symmetric. Now consider the following:

$$\mathbf{M} + \tilde{\mathbf{M}} = (\mathbf{M}_A + \mathbf{M}_S) + (\tilde{\mathbf{M}}_A + \tilde{\mathbf{M}}_S) = (\mathbf{M}_A + \mathbf{M}_S) + (-\mathbf{M}_A + \mathbf{M}_S) = 2\mathbf{M}_S$$

Therefore, for any decomposition of $\mathbf{M}$ into symmetric and antisymmetric matrices, the symmetric part $\mathbf{M}_S$ must satisfy $\mathbf{M}_S = \frac{1}{2}(\mathbf{M} + \tilde{\mathbf{M}})$. By considering $\mathbf{M} - \tilde{\mathbf{M}}$, we can prove similarly that $\mathbf{M}_A = \frac{1}{2}(\mathbf{M} - \tilde{\mathbf{M}})$. Therefore, both $\mathbf{M}_S$ and $\mathbf{M}_A$ are unique.

16.25 ⋆⋆⋆ At a certain point P (which you can choose to be your origin) in a continuous solid, the strain tensor is $\mathbf{E}$. Assume for simplicity that whatever displacements have occurred left P fixed and the neighborhood of P unrotated. **(a)** Show that the x axis near P is stretched by a factor of $(1 + \epsilon_{11})$. **(b)** Hence show that any small volume around P has changed by $dV/V = \mathrm{tr}\,\mathbf{E}$. This shows that any two strains that have the same trace dilate volumes by the same amount. In the decomposition $\mathbf{E} = e\mathbf{1} + \mathbf{E}'$ (16.88), the spherical part $e\mathbf{1}$ changes volumes by the same amount as $\mathbf{E}$ itself, while the deviatoric part $\mathbf{E}'$ doesn't change volumes at all.

(a) Because there is no rotation about P, the strain tensor $\mathbf{E} = \mathbf{D}$, the matrix made up of derivatives $D_{ij} = \partial u_i/\partial r_j$. Now consider a point Q originally located a small displacement $(dx, 0, 0)$ from P. The deformation represented by $\mathbf{E}$ moves Q away from P by an amount $du_1 = (\partial u_1/\partial r_1)dx = \epsilon_{11}dx$. (It may also cause Q to shear in the y and z directions, but that won't affect volumes.) Thus the length that was dx in the x direction has stretched to $dx(1 + \epsilon_{11})$, increasing by a factor $(1 + \epsilon_{11})$.
(b) Similarly, lengths in the y and z directions are stretched by factors $(1 + \epsilon_{22})$ and $(1 + \epsilon_{33})$ respectively. Therefore any small volume around P will increase by a factor

$$(1 + \epsilon_{11})(1 + \epsilon_{22})(1 + \epsilon_{33}) \approx 1 + \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = 1 + \mathrm{tr}\,\mathbf{E},$$

where I have used the smallness of the elements of $\mathbf{E}$ to drop second order terms and higher. Thus a volume that was originally V will increase by $dV = V\mathrm{tr}\,\mathbf{E}$, as claimed.

An alternative and very slick proof uses the divergence theorem:

$$dV = \int_S \mathbf{u} \cdot d\mathbf{A} = \int_V \nabla \cdot \mathbf{u}\, dV \approx V\nabla \cdot \mathbf{u} = V\left(\frac{\partial u_1}{\partial r_1} + \frac{\partial u_2}{\partial r_2} + \frac{\partial u_3}{\partial r_3}\right) = V\mathrm{tr}\,\mathbf{E}$$

where the approximation depends on the smallness of V.

SECTION 16.9 Relation between Stress and Strain: Hooke's Law

16.27 ⋆⋆⋆ Consider a taut wire or rod lying along the x axis. To define Young's modulus YM we apply a pure tension along the axis; that is, a stress with $\sigma_{11} > 0$ and all other $\sigma_{ij} = 0$. **(a)** Use Equation (16.95) to write down the corresponding strain tensor $\mathbf{E}$. **(b)** Argue from the definition (16.55) of Young's modulus to show that $\mathrm{YM} = \sigma_{11}/\epsilon_{11}$. **(c)** Combine these two results to verify the expression (16.100) for YM, showing in particular that $\mathrm{YM} = 9\,\mathrm{BM} \cdot \mathrm{SM}/(3\,\mathrm{BM} + \mathrm{SM})$.

(a) The stress tensor $\boldsymbol{\Sigma}$ is as shown on the left below. From this we can calculate $\mathrm{tr}\,\boldsymbol{\Sigma} = \sigma_{11}$ and the strain tensor $\mathbf{E}$ (16.95) as shown on the right:

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{E} = \frac{1}{3\alpha\beta}\left[3\alpha\mathbf{\Sigma} + (\beta-\alpha)(\operatorname{tr}\mathbf{\Sigma})\,\mathbf{1}\right]$$

$$= \frac{\sigma_{11}}{3\alpha\beta}\begin{bmatrix} 2\alpha+\beta & 0 & 0 \\ 0 & \beta-\alpha & 0 \\ 0 & 0 & \beta-\alpha \end{bmatrix}.$$

(b) In the definition (16.55) of YM, the left side is $dF/A = \sigma_{11}$, and the factor dl/l on the right is ϵ_{11}. Therefore, $\text{YM} = \sigma_{11}/\epsilon_{11}$.
(c) Reading off ϵ_{11} from part (a), we find

$$\text{YM} = \frac{\sigma_{11}}{\epsilon_{11}} = \frac{3\alpha\beta}{2\alpha+\beta} = \frac{9\,\text{BM}\cdot\text{SM}}{3\,\text{BM}+\text{SM}}$$

because, according to (16.97) and (16.99), $\alpha = 3\text{BM}$ and $\beta = 2\text{SM}$.

16.29 ⋆⋆⋆ When we change our coordinate axes, the strain tensor changes in accordance with Equation (15.132), which we can rewrite as $\mathbf{E}_R = \mathbf{R}\mathbf{E}\tilde{\mathbf{R}}$, where $\mathbf{R}$ is the (3×3) orthogonal rotation matrix. Use the property (15.129) of orthogonal matrices to show that $\operatorname{tr}\mathbf{E}_R = \operatorname{tr}\mathbf{E}$; that is, the trace of any tensor is rotationally invariant. Use this result to show that the decomposition $\mathbf{E} = e\mathbf{1} + \mathbf{E}'$ is rotationally invariant, in the sense described below Equation (16.92).

$$\operatorname{tr}(\mathbf{E}_R) = \operatorname{tr}(\mathbf{R}\mathbf{E}\tilde{\mathbf{R}}) = \sum_i (\mathbf{R}\mathbf{E}\tilde{\mathbf{R}})_{ii} = \sum_{i,j,k} R_{ij}E_{jk}\tilde{R}_{ki} = \sum_{i,j,k} \tilde{R}_{ki}R_{ij}E_{jk}$$

$$= \sum_k (\tilde{\mathbf{R}}\mathbf{R}\mathbf{E})_{kk} = \sum_k E_{kk} = \operatorname{tr}(\mathbf{E}). \tag{S16.3}$$

Suppose that we start in the reference frame $\mathcal{S}$, and decompose $\mathbf{E}$ as $\mathbf{E} = e\mathbf{1} + \mathbf{E}'$, where, by definition $e = \frac{1}{3}\operatorname{tr}(\mathbf{E})$. If, now, we make a rotation $\mathbf{R}$ to a new frame, $\mathcal{S}_R$ say, then we find for the new (rotated) strain tensor

$$\mathbf{E}_R = \mathbf{R}\mathbf{E}\tilde{\mathbf{R}} = \mathbf{R}(e\mathbf{1} + \mathbf{E}')\tilde{R} = e\mathbf{1} + \mathbf{R}\mathbf{E}'\tilde{\mathbf{R}}. \tag{S16.4}$$

because $\mathbf{R}\tilde{\mathbf{R}} = \mathbf{1}$. On the other hand, if we start in the frame $\mathcal{S}_R$ and decompose the rotated matrix $\mathbf{E}_R$, we will get

$$\mathbf{E}_R = e_R\mathbf{1} + (\mathbf{E}_R)' \tag{S16.5}$$

where by definition $e_R = \operatorname{tr}(\mathbf{E}_R)$. Now by the result (S16.3), $e_R = e$. Therefore the first terms on the right sides of Eqs. (S16.4) and (S16.5) are the same. It follows that the second terms are also equal, and we've proved that (1) the spherical parts of $\mathbf{E}_R$ and $\mathbf{E}$ are the same and (2) the deviatoric part of $\mathbf{E}_R$ is the result of rotating the deviatoric part of $\mathbf{E}$; that is $(\mathbf{E}_R)' = \mathbf{R}\mathbf{E}'\tilde{\mathbf{R}}$.

SECTION 16.11 Longitudinal and Transverse Waves in a Solid

16.31 ⋆ A seismograph records the signals arriving from a distant earthquake. If the S waves arrive 12 minutes after the P waves, how far away was the earthquake? Use the speeds found in Example 16.8 (page 722).

The times of travel of the two waves are $t_{\text{long}} = d/c_{\text{long}}$ and $t_{\text{tran}} = d/c_{\text{tran}}$, where d is the distance from the quake to the observer. Therefore the time between the arrivals of the two signals is $\Delta t = t_{\text{tran}} - t_{\text{long}} = d(1/c_{\text{tran}} - 1/c_{\text{long}})$, and

$$d = \frac{\Delta t}{1/c_{\text{tran}} - 1/c_{\text{long}}} = \frac{720\text{ s}}{(1/3.0 - 1/5.25)\text{ s/km}} = 5040\text{ km}.$$

SECTION 16.12 Fluids: Description of the Motion

16.33 ⋆ Write down the equation of motion (16.124) as applied to a *static* fluid. Assuming that $\mathbf{g}$ is uniform and ϱ is constant (independent of $\mathbf{r}$), prove the well-known result from introductory physics that the pressure difference between two points $\mathbf{r}_1$ and $\mathbf{r}_2$ is just $\Delta p = \varrho g h$, where h is the vertical difference in elevations of $\mathbf{r}_1$ and $\mathbf{r}_2$.

With $\mathbf{v} = 0$, Eq. (16.124), $\varrho d\mathbf{v}/dt = \varrho \mathbf{g} - \nabla p$ reduces to $\varrho \mathbf{g} = \nabla p$. Integrating both sides gives $\int \varrho \mathbf{g} \cdot d\mathbf{r} = \int \nabla p \cdot d\mathbf{r}$ or $\varrho g \int dz = \int dp$ if we measure z vertically down. Therefore $\varrho g h = \Delta p$.

SECTION 16.13 Waves in a Fluid

16.35 ⋆ A crucial step in showing that the waves in a fluid are necessarily longitudinal was the integral in (16.142). For an arbitrary function $f(\xi)$, with derivative $f'(\xi)$, prove that $\int f'(\mathbf{n} \cdot \mathbf{r} - ct)dt = -f(\mathbf{n} \cdot \mathbf{r} - ct)/c$.

If we make the substitution $\xi = \mathbf{n} \cdot \mathbf{r} - ct$, with $\mathbf{n}$, $\mathbf{r}$, and c all fixed, then $d\xi = -c\,dt$ and

$$\int f'(\mathbf{n} \cdot \mathbf{r} - ct)dt = \frac{-1}{c} \int f'(\xi)d\xi = \frac{-1}{c} f(\xi) = \frac{-1}{c} f(\mathbf{n} \cdot \mathbf{r} - ct.)$$

16.37 ⋆⋆ Show that the intensity I of a sound wave is proportional to the square of the pressure increment p'. To do this consider a small sliver of fluid normal to the direction of propagation with area dA. Write down the rate at which this sliver does work on the fluid just in front of it, then divide by dA to show that the time-average intensity is $\langle I \rangle = \langle p'^2 \rangle / c\varrho_{\text{o}}$.

The force exerted by a sliver of air of area dA on the air just ahead of it is $\mathbf{F} = p\mathbf{n}dA = (p_{\text{o}} + p')\mathbf{n}dA$ where $\mathbf{n}$ is the normal to the sliver, in the direction of propagation. The rate at which this force does work is $\mathbf{F} \cdot \mathbf{v}$ and the intensity I is found by dividing this by the area, $I = \mathbf{F} \cdot \mathbf{v}/dA = (p_{\text{o}} + p')\mathbf{n} \cdot \mathbf{v}$. According to (16.142), $\mathbf{v} = p'\mathbf{n}/c\varrho_{\text{o}}$. Therefore $I = (p_{\text{o}} + p')p'/c\varrho_{\text{o}}$. This gives I as the sum of two terms. When p' varies sinusoidally, the first term averages to zero, and we're left with $\langle I \rangle = \langle p'^2 \rangle / c\varrho_{\text{o}}$, as claimed.

Trigonometric Identities

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi \qquad \cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$

$$\cos\theta\cos\phi = \tfrac{1}{2}[\cos(\theta+\phi) + \cos(\theta-\phi)] \qquad \sin\theta\sin\phi = \tfrac{1}{2}[\cos(\theta-\phi) - \cos(\theta+\phi)]$$

$$\sin\theta\cos\phi = \tfrac{1}{2}[\sin(\theta+\phi) + \sin(\theta-\phi)]$$

$$\cos^2\theta = \tfrac{1}{2}[1+\cos 2\theta] \qquad \sin^2\theta = \tfrac{1}{2}[1-\cos 2\theta]$$

$$\cos\theta + \cos\phi = 2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} \qquad \cos\theta - \cos\phi = 2\sin\frac{\theta+\phi}{2}\sin\frac{\phi-\theta}{2}$$

$$\sin\theta \pm \sin\phi = 2\sin\frac{\theta\pm\phi}{2}\cos\frac{\theta\mp\phi}{2}$$

$$\cos^2\theta + \sin^2\theta = 1 \qquad \sec^2\theta - \tan^2\theta = 1$$

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \text{[Euler's relation]}$$

$$\cos\theta = \tfrac{1}{2}(e^{i\theta} + e^{-i\theta}) \qquad \sin\theta = \tfrac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

Hyperbolic Functions

$$\cosh z = \tfrac{1}{2}(e^z + e^{-z}) = \cos(iz) \qquad \sinh z = \tfrac{1}{2}(e^z - e^{-z}) = -i\sin(iz)$$

$$\tanh z = \frac{\sinh z}{\cosh z} \qquad \text{sech} z = \frac{1}{\cosh z}$$

$$\cosh^2 z - \sinh^2 z = 1 \qquad \text{sech}^2 z + \tanh^2 z = 1$$

Series Expansions

$$f(z) = f(a) + f'(a)(z-a) + \tfrac{1}{2!}f''(a)(z-a)^2 + \tfrac{1}{3!}f'''(a)(z-a)^3 + \cdots \qquad \text{[Taylor's series]}$$

$$e^z = 1 + z + \tfrac{1}{2!}z^2 + \tfrac{1}{3!}z^3 + \cdots \qquad \ln(1+z) = z - \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 - \cdots [\,|z| < 1]$$

$$\cos z = 1 - \tfrac{1}{2!}z^2 + \tfrac{1}{4!}z^4 - \cdots \qquad \sin z = z - \tfrac{1}{3!}z^3 + \tfrac{1}{5!}z^5 - \cdots$$

$$\cosh z = 1 + \tfrac{1}{2!}z^2 + \tfrac{1}{4!}z^4 + \cdots \qquad \sinh z = z + \tfrac{1}{3!}z^3 + \tfrac{1}{5!}z^5 + \cdots$$

$$\tan z = z + \tfrac{1}{3}z^3 + \tfrac{2}{15}z^5 + \cdots [\,|z| < \pi/2] \qquad \tanh z = z - \tfrac{1}{3}z^3 + \tfrac{2}{15}z^5 - \cdots [\,|z| < \pi/2]$$

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{2!}z^2 + \cdots [\,|z| < 1] \qquad \text{[binomial series]}$$

Some Derivatives

$$\frac{d}{dz}\tan z = \sec^2 z \qquad \frac{d}{dz}\tanh z = \operatorname{sech}^2 z$$

$$\frac{d}{dz}\sinh z = \cosh z \qquad \frac{d}{dz}\cosh z = \sinh z$$

Some Integrals

$$\int \frac{dx}{1+x^2} = \arctan x \qquad \int \frac{dx}{1-x^2} = \operatorname{arctanh} x$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x \qquad \int \frac{dx}{\sqrt{1+x^2}} = \operatorname{arcsinh} x$$

$$\int \tan x \, dx = -\ln\cos x \qquad \int \tanh x \, dx = \ln\cosh x$$

$$\int \frac{dx}{x+x^2} = \ln\left(\frac{x}{1+x}\right) \qquad \int \frac{x\,dx}{1+x^2} = \frac{1}{2}\ln(1+x^2)$$

$$\int \frac{dx}{\sqrt{x^2-1}} = \operatorname{arccosh} x \qquad \int \frac{x\,dx}{\sqrt{1+x^2}} = \sqrt{1+x^2}$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \arccos(1/x) \qquad \int \frac{\sqrt{x}\,dx}{\sqrt{1-x}} = \arcsin(\sqrt{x}) - \sqrt{x(1-x)}$$

$$\int \frac{dx}{(1+x^2)^{3/2}} = \frac{x}{(1+x^2)^{1/2}} \qquad \int \ln(x)\,dx = x\ln(x) - x$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-mx^2}} = K(m), \quad \text{complete elliptic integral of first kind}$$

Miscellaneous Data (for use in some end-of-chapter problems)

Solar System

(mass of earth) = 5.97×10^{24} kg
(radius of earth) = 6.38×10^{6} m
(mass of moon) = 7.35×10^{22} kg
(radius of moon) = 1.74×10^{6} m
(mass of sun) = 1.99×10^{30} kg
(radius of sun) = 6.96×10^{8} m
(earth–moon distance) = 3.84×10^{8} m
(earth–sun distance) = 1.50×10^{11} m

Ideal Gases

Avogadro's number, $N_A = 6.02 \times 10^{23}$ particles/mole
Boltzmann's constant, $k = 1.38 \times 10^{-23}$ J/K = 8.62×10^{-5}eV/K
Gas constant, $R = 8.31$ J/(mole·K) = 0.0821 liter·atm/(mole·K)
STP = 0°C and 1 atm
(Volume of 1 mole of gas at STP) = 22.4 liters

Conversion Factors

Area: 1 barn = 10^{-28} m^2
Energy: 1 eV = 1.60×10^{-19} J
1 cal = 4.184 J
Length: 1 inch = 2.54 cm
1 mile = 1609 m
Mass: 1 u (atomic mass unit) = 1.66×10^{-27} kg = 931.5 MeV/c^2
1 lb (mass) = 0.454 kg
1 MeV/$c^2 = 1.074 \times 10^{-3}$ u = 1.783×10^{-30} kg
Momentum: 1 MeV/$c = 5.34 \times 10^{-22}$ kg·m/s

A Few More Constants

Coulomb force constant, $k = 1/(4\pi\epsilon_o) = 8.99 \times 10^{9}$ N·m^2/C^2
Gravitational constant, $G = 6.67 \times 10^{-11}$ N·m^2/kg^2
Planck's constants: $h = 6.63 \times 10^{-34}$ J·s and $\hbar = 1.05 \times 10^{-34}$ J·s
Speed of light, $c = 3.00 \times 10^{8}$ m/s
Vacuum permeability, $\mu_o = 4\pi \times 10^{-7}$ N/A^2
Vacuum permittivity, $\epsilon_o = 8.85 \times 10^{-12}$ C^2/(N·m^2)

Vector Identities

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \qquad [BAC - CAB \text{ rule}]$$

Vector Calculus

$$
\begin{aligned}
\nabla f &= \hat{\mathbf{x}}\,\frac{\partial f}{\partial x} + \hat{\mathbf{y}}\,\frac{\partial f}{\partial y} + \hat{\mathbf{z}}\,\frac{\partial f}{\partial z} && \text{[Cartesian]} \\
&= \hat{\mathbf{r}}\,\frac{\partial f}{\partial r} + \hat{\boldsymbol{\theta}}\,\frac{1}{r}\,\frac{\partial f}{\partial \theta} + \hat{\boldsymbol{\phi}}\,\frac{1}{r\sin\theta}\,\frac{\partial f}{\partial \phi} && \text{[spherical polars]} \\
&= \hat{\boldsymbol{\rho}}\,\frac{\partial f}{\partial \rho} + \hat{\boldsymbol{\phi}}\,\frac{1}{\rho}\,\frac{\partial f}{\partial \phi} + \hat{\mathbf{z}}\,\frac{\partial f}{\partial z} && \text{[cylindrical polars]}
\end{aligned}
$$

$$
\begin{aligned}
\nabla \times \mathbf{A} &= \hat{\mathbf{x}}\left(\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right) + \hat{\mathbf{y}}\left(\frac{\partial}{\partial z}A_x - \frac{\partial}{\partial x}A_z\right) \\
&\quad + \hat{\mathbf{z}}\left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right) && \text{[Cartesian]} \\
&= \hat{\mathbf{r}}\,\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, A_\phi) - \frac{\partial}{\partial \phi}A_\theta\right] + \hat{\boldsymbol{\theta}}\left[\frac{1}{r\sin\theta}\,\frac{\partial}{\partial \phi}A_r - \frac{1}{r}\,\frac{\partial}{\partial r}(rA_\phi)\right] \\
&\quad + \hat{\boldsymbol{\phi}}\,\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial \theta}A_r\right] && \text{[spherical polar]} \\
&= \hat{\boldsymbol{\rho}}\left[\frac{1}{\rho}\,\frac{\partial}{\partial \phi}A_z - \frac{\partial}{\partial z}A_\phi\right] + \hat{\boldsymbol{\phi}}\left[\frac{\partial}{\partial z}A_\rho - \frac{\partial}{\partial \rho}A_z\right] \\
&\quad + \hat{\mathbf{z}}\,\frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial}{\partial \phi}A_\rho\right] && \text{[cylindrical polar]}
\end{aligned}
$$

$$
\begin{aligned}
\nabla \cdot \mathbf{A} &= \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z && \text{[Cartesian]} \\
&= \frac{1}{r^2}\,\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\,\frac{\partial}{\partial \theta}(\sin\theta\, A_\theta) + \frac{1}{r\sin\theta}\,\frac{\partial}{\partial \phi}A_\phi && \text{[spherical polars]} \\
&= \frac{1}{\rho}\,\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\,\frac{\partial}{\partial \phi}A_\phi + \frac{\partial}{\partial z}A_z && \text{[cylindrical polars]}
\end{aligned}
$$

$$
\begin{aligned}
\nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} && \text{[Cartesian]} \\
&= \frac{1}{r}\,\frac{\partial^2}{\partial r^2}(rf) + \frac{1}{r^2\sin\theta}\,\frac{\partial}{\partial \theta}\left(\sin\theta\,\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\,\frac{\partial^2 f}{\partial \phi^2} && \text{[spherical polars]} \\
&= \frac{1}{\rho}\,\frac{\partial}{\partial \rho}\left(\rho\,\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\,\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} && \text{[cylindrical polars]}
\end{aligned}
$$